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INVESTIGATION OF STRIPPING REACTIONS ON CHROMIUM ISOTOPES

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The angular distributions of protons from the (d, p) reactions on the chromium isotopes Cr^{50} , Cr^{52} , Cr^{53} , and Cr^{54} are studied using 13.6-MeV protons, and are compared with the simple Butler theory of stripping reactions. The values of l_n for captured neutrons are obtained for the ground state and several excited states of the final nuclei. Some conclusions are reached regarding possible spins and parities of states of the final nuclei Cr^{51} , Cr^{53} , Cr^{54} , and Cr^{55} .

IN the present work we continue our earlier investigations^[1] of stripping reactions in targets containing single isotopes with $A \approx 60$. We have now studied the angular distributions from stripping reactions on the four chromium isotopes Cr^{50} , Cr^{52} , Cr^{53} , and Cr^{54} .

Other workers have studied (d, p) reactions using deuterons up to 21 MeV.^[2-5] However, except in^[5], all targets were made of natural chromium. Thus in one of the first investigations Schiffer et al^[2] studied the gross structure of the proton spectrum from the (d, p) reaction on natural chromium. The angular distributions of protons were compared with the simple Butler theory.

Elwyn and Shull^[5] investigated the proton angular distributions from (d, p) on Cr^{52} and Cr^{53} using 10-MeV deuterons. A comparison with the theory yielded values of the orbital angular momentum l_n for neutrons captured into different states of the final nuclei.

Šlaus^[3] studied the mechanism of the $\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$ reaction with 3.29- and 4.4-MeV deuterons. Although these deuteron energies are far below the Coulomb barrier, the experimental data indicate that direct interactions play an important part.

El Bedewi and Tadros^[4] employed magnetic analysis to investigate $\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$ with 8.67-MeV deuterons. They compared the angular distributions of nine proton groups with Butler's theory and obtained possible spins and parities for states of the final nucleus Cr^{53} .

The (d, p) reactions on Cr^{50} and Cr^{54} have not been investigated previously. It was therefore of interest to investigate the angular distributions of protons from this reaction on all four chromium isotopes.

1. MEASUREMENT TECHNIQUE

Our work was done at the cyclotron of the Institute of Physics of the Academy of Sciences of the

Table I

Tar- get	Thick- ness, mg/cm ²	Content, %			
		Cr ⁵⁴	Cr ⁵²	Cr ⁵³	Cr ⁵⁴
Cr ⁵⁰	2.895	87.7	11.1	0.9	0.3
Cr ⁵²	3.211	0.2	99.0	0.7	0.1
Cr ⁵³	3.790	0.2	13.8	84.3	1.7
Cr ⁵⁴	2.895	0.4	10.9	10.1	78.6

Ukrainian S.S.R., which furnished a 13.6-MeV deuteron beam. The apparatus and measurement technique have been described in [1]. The targets in the form of free thin films 20 mm in diameter were enriched in the chromium isotope to be investigated. Table I gives the target thicknesses and isotopic compositions.

The (d, p) proton angular distributions were obtained at c.m. angles from 7.5° to 100° and are compared with theoretical calculations based on the simple Butler theory.^[6]

The indicated experimental errors in the angular distributions were obtained by averaging several measurements, and include statistical errors not greater than 3%, errors of proton-group resolution, errors associated with inaccurate angular positioning, etc. All these errors pertain to measurements of the angular distributions in relative units. The error in the absolute value of the differential (d, p) cross section was $\pm 50\%$.

2. RESULTS AND DISCUSSION

Figure 1 shows the (d, p) proton spectra resolved into several groups whose angular distributions were obtained. The different proton groups

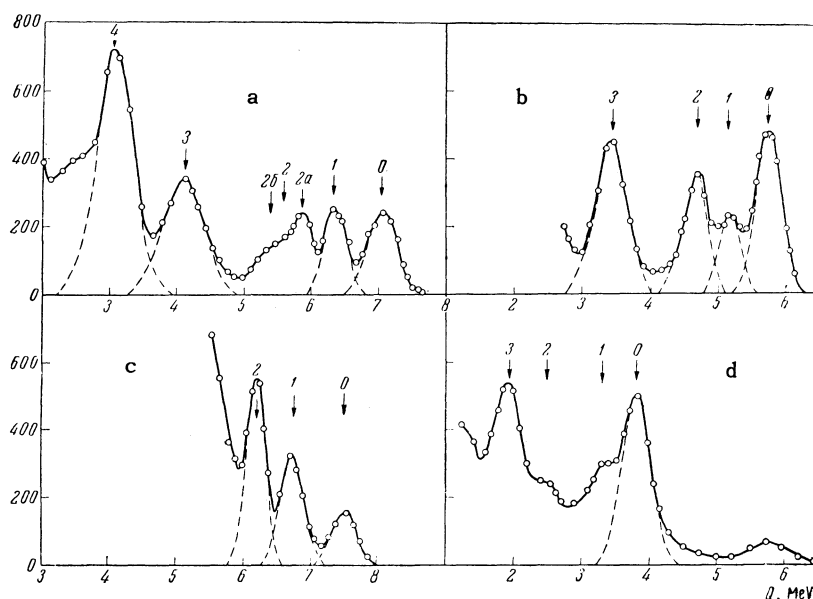
have been designated with the numerical indices 0, 1, 2 etc. Since the inadequate resolving power of the spectrometer did not always permit good resolution of the proton groups, in determining their areas we assumed Gaussian shapes (the dashed curves) of the spectral lines. Consequently, although in certain instances some proton groups could not be resolved easily, very good agreement was obtained between the angular distributions from different runs in which the proton groups were resolved independently. The angular distributions of the corresponding proton groups are therefore quite reliable.

We shall now discuss briefly the separate results for each reaction.

a) $\text{Cr}^{50}(\text{d}, \text{p})\text{Cr}^{51}$. Figure 2 shows the angular distributions of protons. The angular distributions for the ground state (Fig. 2a) and for the first excited state (Fig. 2b) correspond to separate levels in good agreement with the theoretical curves. Table II gives the characteristics of the levels obtained from an analysis of these data. On the shell model the most probable spin and parity are $7/2^-$ for the ground state and $3/2^-$ for the first excited state. According to the Cr^{51} level scheme^[8] the first excited level at 0.75 MeV is followed by a group of excited levels at 1.17, 1.42, and 1.53 MeV. In our experimental work we were unable to resolve these levels separately; therefore we obtained a combined angular distribution for all three levels.

In Fig. 1a this proton group is designated by the number 2 and is seen to consist of at least two groups (2a and 2b) corresponding to the approximate excitation energies 1.2 and 1.5 MeV, respectively. The corresponding angular distribution is shown in Fig. 2. The spectra obtained at different

FIG. 1. Proton spectra from the (d, p) reaction on chromium isotopes: a — $\text{Cr}^{50}(\text{d}, \text{p})\text{Cr}^{51}$, $\theta = 30^\circ$; b — $\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$, $\theta = 30^\circ$; c — $\text{Cr}^{53}(\text{d}, \text{p})\text{Cr}^{54}$, $\theta = 30^\circ$; d — $\text{Cr}^{54}(\text{d}, \text{p})\text{Cr}^{55}$, $\theta = 40^\circ$.



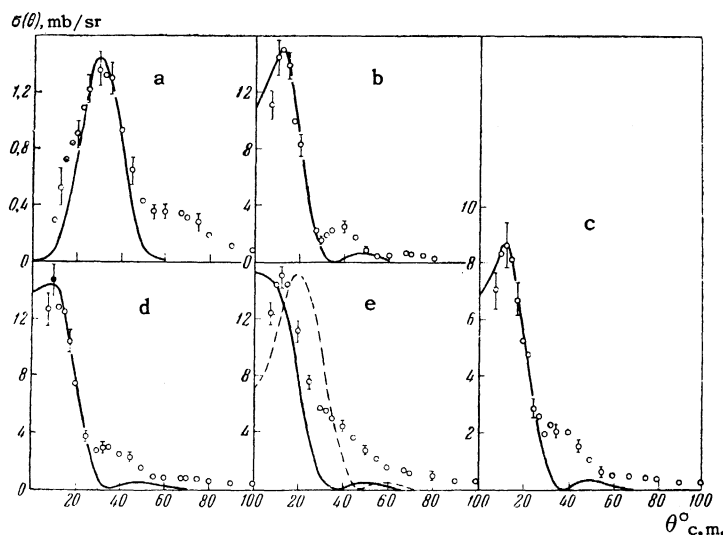


FIG. 2. Angular distributions of protons from $\text{Cr}^{50}(\text{d}, \text{p})\text{Cr}^{51}$. $\circ$ — experimental points accompanied by Butler theoretical curves. a — $l_n = 3$, $r_0 = 6.19 \times 10^{-13}$ cm; b — $l_n = 1$, $r_0 = 5.8 \times 10^{-13}$ cm; c and d — $l_n = 1$, $r_0 = 5.5 \times 10^{-13}$ cm; e — $l_n = 1$, $r_0 = 5.5 \times 10^{-13}$ cm (continuous curve) and $l_n = 2$, $r_0 = 5.5 \times 10^{-13}$ cm (dashed curve).

Table II

Reaction	Q, MeV	Excitation energy, MeV	Proposed value of l_n	Radius r_0 , 10^{-13} cm	Possible spins and parities	Differential cross section at peak, mb/sr	Reduced width $(2J+1)\theta^2$
$\text{Cr}^{50}(\text{d}, \text{p})\text{Cr}^{51}$	7.08	0	3	6.19	$5/2^-, 7/2^-$	1.36	0.054
	6.33	0.75	1	5.8	$1/2^-, 3/2^-$	15.05	0.076
	5.70	1.17	1	5.5	$1/2^-, 3/2^-$	8.57	0.042
		1.42					
		1.53					
$\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$	4.16	2.92	1	5.5	$1/2^-, 3/2^-$	14.69	0.055
	3.11	3.97	1	5.5	$1/2^-, 3/2^-$	14.95	0.049
	5.74	0	1	5.5	$1/2^-, 3/2^-$	24.90	0.122
	5.16	0.58	1	5.5	$1/2^-, 3/2^-$	9.38	0.040
	4.73	1.00	3	6.0	$5/2^-, 7/2^-$	1.92	0.054
$\text{Cr}^{53}(\text{d}, \text{p})\text{Cr}^{54}$	3.41	2.33	1	5.5	$1/2^-, 3/2^-$	18.55	0.060
	7.55	0	1	6.28	$0^+, 1^+, 2^+, 3^+$	1.31	0.031
	6.69	0.81	1	5.8	$0^+, 1^+, 2^+, 3^+$	3.76	0.079
	6.26	1.23	1	5.5	$0^+, 1^+, 2^+, 3^+$	3.38	0.072
$\text{Cr}^{54}(\text{d}, \text{p})\text{Cr}^{55}$	3.8	0	1	5.5	$1/2^-, 3/2^-$	24.63	0.084

angles indicate that the given proton groups resulted from the capture of neutrons with different values of l_n . Thus the proton group 2b is very well resolved and gives the highest yield at about 12° , whereas the group 2a is most strongly resolved at 30° . It can therefore be assumed that the group 2a results from a transition with $l_n = 3$, and the group 2b from a transition with $l_n = 1$. This to some degree confirms our angular distribution for both groups taken together as shown in Fig. 2c, where we also have the theoretical curve for $l_n = 1$ and $r_0 = 5.5 \times 10^{-13}$ cm.

The experimental angular distribution is seen to agree with theory for $l_n = 1$. However, there is no perceptible experimental minimum in the range $30-35^\circ$, as is usually found in the case of $l_n = 1$ (Fig. 2b). This can be due to the contribution of

the second proton group (2b) resulting from a transition with $l_n = 3$.

For the same reaction we obtained two more angular distributions of the proton groups 3 and 4 (Fig. 2d, e) corresponding to excitation energies of the order 2.92 and 3.97 MeV. Since a Cr^{51} level scheme is lacking for this energy region, we cannot state that the corresponding proton groups result from neutron capture into separate levels.

Figure 2 shows the angular distribution for the 2.92-MeV state and the Butler theoretical curve for $l_n = 1$ and $r_0 = 5.5 \times 10^{-13}$ cm. Taking the indicated value $l_n = 1$, we can assign $1/2^-$ or $3/2^-$ to this state.

The angular distribution for the 3.97-MeV state is shown in Fig. 2e, together with the Butler theoretical curves for $l_n = 1$ and 2. The experimental

points are seen to lie between the theoretical curves and can hardly be reconciled with either one. In [9] similar difficulties have been pointed out regarding the choice of l_n for states with low Q in analyzing stripping reactions on nuclei of medium atomic weight with 10-MeV deuterons. This is associated with the fact that Butler's theory disregards both the Coulomb and the nuclear reactions between deuterons and nuclear protons. By using the distorted-wave theory [10] in calculating theoretical angular distributions we obtain considerably better agreement with experiment and the elimination of ambiguities regarding the selection of l_n . Thus, as Scott [9] has shown, when the experimental points lie between Butler curves for two adjacent values of l_n , the theoretical curve based on the distorted-wave theory for the lower value of l_n agrees well with the experimental distribution.

We can take $l_n = 1$ for the investigated 3.97-MeV state on the basis of the foregoing discussion. Then the possible spins and parities will be $\frac{1}{2}^-$ and $\frac{3}{2}^-$.

b) $\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$. The angular distributions of four proton groups from this reaction are shown in Fig. 3 along with the corresponding theoretical curves. Table II gives the orbital angular momenta of neutrons captured into the corresponding states, and the spins and parities of these states.

On the shell model the most probable spin and parity of the Cr^{53} ground state will be $\frac{3}{2}^-$, with $\frac{5}{2}^-$ for the second excited state (1.0 MeV). These results are confirmed by data in [4,5].

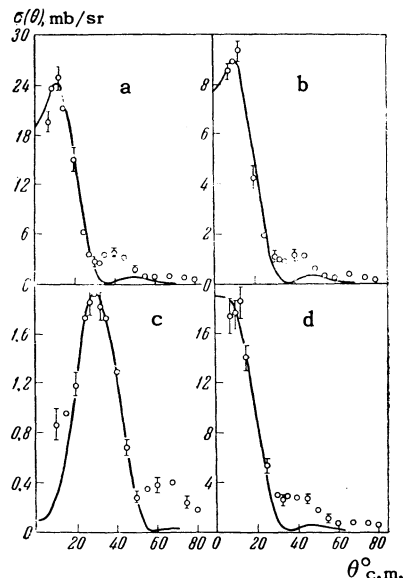
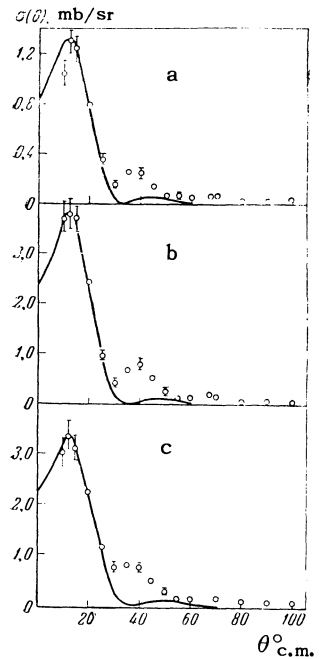


FIG. 3. Angular distributions of proton groups from $\text{Cr}^{52}(\text{d}, \text{p})\text{Cr}^{53}$. $\circ$ - experimental points; continuous lines - Butler theoretical curves. a, b, and d - $l_n = 1$, $r_0 = 5.5 \times 10^{-13}$ cm; c - $l_n = 3$, $r_0 = 6 \times 10^{-13}$ cm.

FIG. 4. Angular distributions of protons from $\text{Cr}^{53}(\text{d}, \text{p})\text{Cr}^{54}$. $\circ$ - experimental points; continuous lines - Butler theoretical curves. a - $l_n = 1$, $r_0 = 6.28 \times 10^{-13}$ cm; b - $l_n = 1$, $r_0 = 5.8 \times 10^{-13}$ cm; c - $l_n = 1$, $r_0 = 5.5 \times 10^{-13}$ cm.



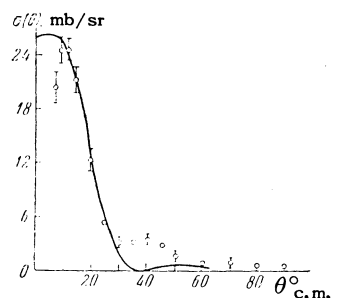
For the first excited Cr^{53} state at 0.58 MeV the possible spin and parity will be $\frac{1}{2}^-$ or $\frac{3}{2}^-$. Wilkinson and Sheline [11] have identified this level as a $2p_{1/2}$ state on the basis of relative γ -ray intensities in neutron capture.

c) $\text{Cr}^{53}(\text{d}, \text{p})\text{Cr}^{54}$. Figure 4 shows the proton angular distributions from this reaction and the Butler theoretical curves; $l_n = 1$ was obtained for all angular distributions. Since the target is a spin- $\frac{3}{2}^-$ even-odd nucleus, the investigated levels must have positive parity and the possible spins are 0, 1, 2, and 3.

The final Cr^{54} nucleus has an even mass and must therefore have a 0^+ ground state. The most probable assignment for the first excited state of Cr^{54} is 2^+ . Our values of l_n for all three states agree with those given in [7].

d) $\text{Cr}^{54}(\text{d}, \text{p})\text{Cr}^{55}$. Figure 1d shows the proton spectrum from this reaction. Only two groups (0 and 3) have appreciable intensities; the other two groups (1 and 2) are very weak at the given angle. At other angles the intensity ratio of these groups changes. Since it is very difficult to discriminate all these proton groups and the level

FIG. 5. Angular distribution of protons from $\text{Cr}^{54}(\text{d}, \text{p})\text{Cr}^{55}$. $\circ$ - experimental points; continuous line - Butler theoretical curve for $l_n = 1$ and $r_0 = 5.5 \times 10^{-13}$ cm.



scheme of Cr^{55} is entirely unknown, we plotted the angular distribution only for the ground-state group; this is shown in Fig. 5 in comparison with the Butler theoretical curve for $l_n = 1$. Since, according to the shell model, in the Cr^{55} ground state the 31st neutron should be captured into the $2p_{3/2}$ shell, the most probable spin and parity assignment for this state will be $3/2^-$, in good agreement with [8].

Regarding the proton group 1 we can only comment about the angular dependence of its intensity. At small angles the group is very weak and almost entirely imperceptible in the tail of the intense ground-state group. The intensity of group 1 rises with increase of the angle, and at 30° is comparable to the group 0 for the ground state. Its intensity again diminishes with further increase of the angle. It is thus suggested that the group results from the capture of neutrons with $l_n = 3$. This agrees with the shell model, according to which the $1f_{5/2}$ level should lie above the $2p_{3/2}$ level. However, final conclusions regarding this state will await an angular distribution based on measurements in which these groups can be resolved.

On the basis of the measured absolute values of the differential cross sections for the investigated (d, p) reactions and the analysis of the angular distributions we employed Butler's theory to calculate the reduced widths $(2J + 1)\theta^2$, where J is the spin of the final-nucleus state. An equation in [12] was used to calculate $(2J + 1)\theta^2$. Table II gives the reduced widths and other characteristics of states in the investigated nuclei.

In conclusion we wish to thank A. P. Klyucharev for providing the enriched targets, and also the crew that was responsible for the uninterrupted operation of the cyclotron.

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MEASUREMENT OF THE GYROMAGNETIC RATIO OF THE W^{182} NUCLEUS IN THE FIRST EXCITED STATE

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The gyromagnetic ratio of the 100-keV level of the W^{182} nucleus is measured by means of the perturbation of the $\gamma\gamma$ correlation of the 229—100 keV cascade by an external magnetic field of 35,000 gauss. The coefficients of the "unperturbed" correlation functions are $A_2 = 0.087 \pm 0.008$ and $A_2 = 0.108 \pm 0.008$ for a solid and liquid source respectively. The angular shifts of the correlation function are $4^\circ 10'$ for the solid and $5^\circ 53'$ for the liquid source; this corresponds to $g = 0.247 \pm 0.037$ and $g = 0.323 \pm 0.048$. The mean value of the gyromagnetic ratio for the two measurements is $g = 0.285 \pm 0.042$.

PRESENT-DAY nuclear models have reached such a stage of development that further refinement requires not only knowledge of the alpha and beta decay probabilities with the excitation of collective motion, knowledge of the excitation of Coulomb rotational levels etc, but also knowledge of a number of other quantities sensitive to specific features of the nuclei. Such quantities are, in the first place, the moments of inertia and the magnetic moments (and consequently the gyromagnetic ratios) of nuclei in the ground and excited states.

New nuclear models have been worked out in recent years which are modifications of the shell and collective models. Thus, in the theory of nuclei lacking axial symmetry proposed by Davydov and Filippov^[1] the coupling is considered between the collective and single-particle motions of the nucleons in the nucleus, whereas it had been hitherto assumed that there is no interaction whatever between these two forms of motion. Migdal,^[2] Beliaev^[3], and Solov'ev^[4] have applied the methods of the Bogolyubov theory of superfluidity to the description of the motion of the nucleons in the nucleus.

The interesting results of the latter models require experimental data which would be a check on the conclusions of the theory. Therefore, experiments connected with a direct determination of nuclear parameters sensitive to specific assumptions of the models, such as magnetic moments and gyromagnetic ratios of the ground and excited states, take on special significance.

The majority of the presently known gyromagnetic ratios and nuclear magnetic moments has been calculated from the experimentally measured

gamma-transition probabilities in nuclei.^[5-7] However, data indirectly obtained to some extent lost their value because it had been assumed in calculating the gyromagnetic ratios that the single-particle and collective motions of the nucleons in the nucleus were completely separated, an assumption which was merely a rough approximation.^[1]

In this connection interest was aroused by the direct method of measuring gyromagnetic ratios of nuclei in the excited state, proposed by Brady and Deutsch.^[8] The theory of this method was worked out by Alder^[9,10] and Lloyd,^[11] and in 1953 the Zürich group first measured the gyromagnetic ratio of the first excited state of the Cd^{111} nucleus.^[12] Subsequently a number of original papers were published, and also several reviews devoted to direct methods of measuring gyromagnetic ratios of excited nuclei.^[13,14]

The following two methods exist for determining the gyromagnetic ratio of nuclei in the excited state: the method of $\gamma\gamma$ correlation and the method of Coulomb excitation. Both methods employ one and the same physical phenomenon—the interaction of the nuclear magnetic moment with an external magnetic field; however, in approach they differ from each other considerably. In both instances the precession of the nuclear magnetic moment in the external field leads to a distortion of the angular distribution of the gamma quanta (of the correlation function of the $\gamma\gamma$ cascade). Let us consider a more general case of the effect of an external magnetic field on the correlation function of the $\gamma\gamma$ cascade.

There exists a definite correlation between the directions of emission of two successive gamma

quanta which form a cascade by means of which the excited nucleus is de-excited. This means that if the counting rates of the time coincidences of quanta γ_1 and γ_2 are measured, a certain distribution is obtained which depends on the angle between the emission directions of γ_1 and γ_2 , i.e., a correlation function $W(\theta)$. The latter can be represented in the form

$$W(\theta) = \sum_k A_k P_k(\cos \theta) = \sum_k a_k \cos k\theta, \quad (1)$$

where A_k and a_k are anisotropy coefficients depending on the spins of the ground and excited state and on the multipolarity of the gamma transitions. P_k are the even Legendre polynomials.

If a magnetic field is applied perpendicular to the plane in which the gamma quanta are registered, then during the time between the instants of emission of quanta γ_1 and γ_2 the nucleus will on account of the precession turn through some angle $\Delta\theta = \omega\tau$ (τ is the mean lifetime of the nucleus in the excited state, ω is the Larmor precession frequency) and the angle between the directions of emission of γ_1 and γ_2 will become $\theta + \Delta\theta$.

The correlation function perturbed by the magnetic field can be readily obtained by integrating the expression

$$W(\theta, H) = \int_{\tau_1}^{\tau_2} W[(\theta + \Delta\theta), H = 0] e^{-t/\tau} dt. \quad (2)$$

The integration is carried out under the assumption that $\tau \ll \tau_R$ (τ_R is the resolving time of the coincidence circuit), and under the assumption that no terms with $k > 2$ enter into expansion (1). Integrating (2), we obtain

$$W(\theta, H) = \sum_k \frac{a_k}{\sqrt{1 + (k\omega\tau)^2}} \cos k(\theta + \Delta\theta_k), \quad (3)^*$$

$$\Delta\theta_k = \frac{1}{k} \arctg(k\omega\tau),$$

the Larmor precession frequency being

$$\omega = \mu H / I\hbar = gH\mu_{\text{nuc}} / \hbar, \quad (4)$$

where H is the magnetic field intensity, I is the spin of the investigated state, g is the gyromagnetic ratio of this state, μ is the nuclear magnetic moment, and μ_{nuc} is the nuclear magneton.

From (3) it can be seen that in a magnetic field the angular distribution function has been shifted by an angle $\Delta\theta$, and the anisotropy of the distribution has decreased; by measuring the azimuthal shift or the decrease in the anisotropy of the cor-

relation function, it is possible to determine the value of ω , from which it is easy to calculate the gyromagnetic ratio g .

The Coulomb excitation method, while also employing the precession of the nuclear magnetic moment in an external magnetic field, differs from the $\gamma\gamma$ -correlation method in the way the investigated nucleus is excited and in the method of registering its precession. In this case the nucleus is excited by electromagnetic interaction with an incident charged particle. Since a preferred direction exists in this case (the point from which the measurement of the precession angle is started and which coincides with the direction of the charged-particle beam), in order to observe the azimuthal shift of the angular distribution and the weakening of its anisotropy in an external magnetic field it is sufficient to measure the angular distribution of the gamma quanta corresponding to transition from the investigated state to the ground state. The value of the gyromagnetic ratio can be found from relations (3) which remain valid in this case.

A number of circumstances complicate greatly the experiments for measuring the gyromagnetic ratios of excited nuclei. In a real source of gamma quanta there always exist local electrical and magnetic fields which themselves perturb the angular distributions of gamma quanta, and whose effect it is very difficult to take into account. On the other hand, knowledge of the magnitude of the perturbation of the angular distribution caused by the interaction of the local fields with the magnetic moment of the nucleus is essential, since the character of the perturbations introduced by the local fields and by the applied magnetic field is identical.

Recently more or less successful attempts have been made to take these "internal" fields into account. The action of the electrical fields is taken into account by introducing so-called "attenuation coefficients" G_k , whose magnitude depends on the chemical composition and the state of aggregation of the source (target) material; to account for the interaction of the local magnetic fields a certain effective field $H_{\text{eff}} = \beta H$ is introduced, where β is a correction factor related to the paramagnetic properties of the material of the gamma source.^[15]

The methodical difficulties connected, on the one hand, with the registration of coincidences (the increase of the background of false coincidences for $\tau_R \geq 10^{-6}$ to an extent that the number of false coincidences begins to be larger than the number of true coincidences), and also the difficulties in obtaining magnetic fields with intensities exceeding 40,000 G, and the "equalizing" action of the mag-

* $\arctg = \tan^{-1}$.

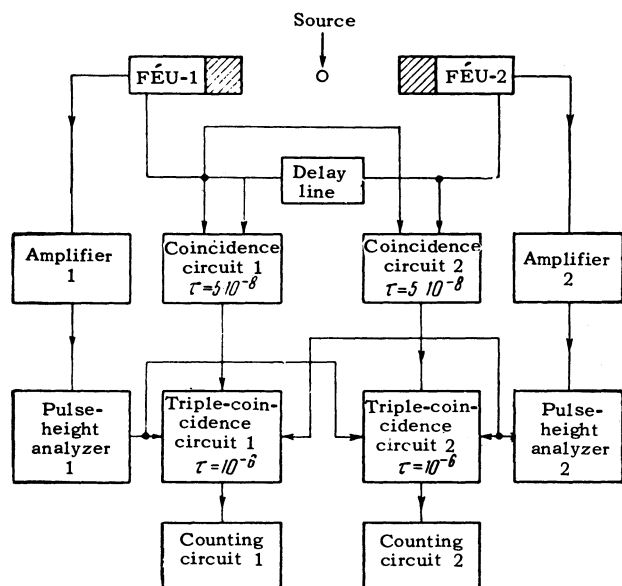


FIG. 1. Block diagram of the electronics. Coincidence circuit 1 registers random coincidences, 2 registers the total number of coincidences.

netic field on the angular distribution of gamma quanta, on the other hand, limit the above methods of measuring the gyromagnetic ratios of excited nuclei; reliable measurements of gyromagnetic ratios can thus be obtained for excited states whose mean lifetimes are within the limits of 10^{-7} – 10^{-10} sec, provided the anisotropy of the corresponding angular distribution is sufficiently large. (The upper limit of the lifetime of the excited state of 10^{-7} sec apparently does not extend entirely to the Coulomb-excitation method.)

We have built a device which permits us to measure the gyromagnetic ratios of nuclei in the excited state both by the method of Coulomb excitation and also by the $\gamma\gamma$ correlation. The main components of the device are: a two-channel goniometer, electronic registration apparatus, and an electromagnet producing in the gap a magnetic field of 35,000 G. The magnet core is made of "Armco" iron. The core gap can be varied from 0 to 30 mm. The gamma detectors consist of 30×15 mm NaI (Tl) scintillators with FÉU-42 photomultipliers. In order to weaken the effect of the magnetic field on the photomultipliers a three-layer magnetic screen is used, and 200-mm organic-glass light guides make it possible to remove the photomultipliers outside the limits of the stray field, retaining at the same time a sufficient gamma-detector count.

The gamma detector pulses enter into a fast-slow coincidence circuit whose block diagram is shown in Fig. 1. The "fast" coincidence resolution time is $2\tau_R = 10^{-7}$ sec. In the "slow" chan-

nels the pulses undergo amplification and pulse-height selection with the aid of linear amplifiers and single-channel pulse-height analyzers. The position of the discriminator "window" and its width are chosen such that only pulses corresponding to gamma quanta producing the investigated cascade (in the case of Coulomb excitation gamma quanta corresponding to a transition from the investigated level to the ground state) enter the "slow" coincidence cell ($\tau_R = 10^{-6}$ sec). Very strict demands are made on the pulse-height analyzers used in this work. We have, therefore, employed specially constructed discriminators which assure the stability of the position and width of the window within 0.03 V under a load of $(2-3) \times 10^4$ pulse/sec.

The gyromagnetic ratio of the first excited state of the even-even W^{182} nucleus was measured by the $\gamma\gamma$ correlation method. Natural metallic tantalum neutron-irradiated in the water-moderated water-cooled reactor of the Physics Institute of the Ukrainian Academy of Sciences served as the gamma source.

In view of the fact that the gamma spectrum appearing in the decay of Ta^{182} is very complex, the form of the correlation function is distorted on account of the considerable number of false coincidences due to "outside" gamma transitions with close energy ($E_\gamma = 222$ keV).

We measured the correlation function of the 229–100 keV cascade which for a solid polycrystalline source turned out to be $W(\theta) = 1 + (0.016 \pm 0.002) \cos 2\theta$ (without subtracting the background due to false coincidences). From a comparison of the relative intensities of the corresponding gamma transitions in the decay scheme of Ta^{182} [16] we could estimate the isotropic background of false coincidences from disturbing gamma transitions and obtain a correlation function of the 229–100 keV cascade in the form $W(\theta) = 1 + (0.087 \pm 0.009) P_2(\cos \theta)$. From a comparison of the experimentally obtained anisotropy with the coefficient A_2 given by the $\gamma\gamma$ -correlation theory, the value of the attenuation coefficient was calculated to be $G_2 = 0.82$.

The gyromagnetic ratio of the 100-keV excited state of W^{182} was measured from the shift of the correlation function in a 35,000-G magnetic field. The results of the measurement are shown in Fig. 2. Reduction of the obtained data by the least-squares method showed that the shift of the correlation function amounts to $\Delta\theta = 4^\circ 10'$. The value of the gyromagnetic ratio corresponding to this shift is $g = 0.247 \pm 0.037$.

Supplementary note (December 12, 1962). A sec-

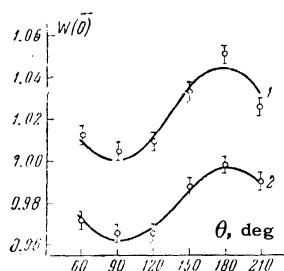


FIG. 2. Measurement results: curve 1 corresponds to the unperturbed correlation function of the 229–100 keV cascade, curve 2 corresponds to the correlation function of the same cascade in a magnetic field $H = 35,000$ G.

ond series of measurements was carried out using a liquid source. Activated metallic tantalum was dissolved in a mixture of equal amounts of hydrofluoric and nitric acid.

In the absence of the magnetic field the unperturbed correlation function of the 229–100 keV cascade turned out to be $W(\theta) = 1 + (0.0205 \pm 0.0015) \cos 2\theta$ or, after subtracting the background, $W(\theta) = 1 + (0.108 \pm 0.008) P_2 \cos \theta$. Comparison of the obtained coefficient A_2 with the anisotropy coefficient of the correlation function of the $4^+ - 2^+ - 0^+$ cascade shows that there is practically no weakening of the correlation by the internal fields of the source.

The angular shift of the correlation function measured in a 35,000-G magnetic field amounted to $5^\circ 53'$, corresponding to a gyromagnetic ratio $g = 0.323 \pm 0.048$.

The values of the gyromagnetic ratio of the first excited state of W^{182} obtained in both measurements coincide within the experimental errors, and it will apparently be sensible to use as the result the average of the two measurements of the gyromagnetic ratio $g = 0.285 \pm 0.042$. This value is in agreement with the calculated value $g = 0.26$ obtained on the basis of the superfluid model.^[17]

The authors consider it their pleasant duty to express their gratitude to A. P. Klyucharev and V. A. Lutsik (Physico-technical Institute, Academy of Sciences, Ukr. S.S.R.) for assistance and interest in the work, and also to I. F. Barchuk and D. T.

Pilipets (Physics Institute, Academy of Sciences, Ukr. S.S.R.) for help in preparing the necessary gamma sources.

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ANGULAR CORRELATION OF THE γ CASCADE DUE TO THE 440–337 keV TRANSITION IN Sm^{150}

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The angular correlation of the γ cascade emitted by the excited Sm^{150} nucleus in the 440–337 keV transition was measured at 90° , 135° , and 180° . An analysis of the experimental data shows that the spin of the second level in Sm^{150} is 4.

THE $_{62}\text{Sm}^{150}$ nucleus is at the boundary of the region of strongly deformed nuclei, and hence the low-lying excited states are either rotational or vibrational. The ratio of the energies of the second and first excited levels is $E_{II}/E_I = 2.3$ ($E_{II} = 777$ keV, $E_I = 337$ keV) and not 3.3 as in the case of rotational levels; it can therefore be assumed that the spectrum of the ground state is vibrational. If this is so, then the spin of the second level of this nucleus is either 0, 2, or 4. The data reported thus far in the literature are ambiguous. Thus Rosler and Fenstermacher gave the value 3 and Draper^[2] gave the values 2, 3, or 4. In the present article we report the results obtained from the angular-correlation measurements of 440- and 337-keV γ rays emitted by Sm^{150} in the transitions $E_{II} - E_I - E_0$.

The experiment was carried out with a neutron beam from one of the horizontal channels of the VVR-S-2 reactor. The neutrons first passed through a quartz filter 10 cm long to increase the amount of thermal neutrons and then through a CB_4 and lead collimator. This gave a beam diameter of 15 mm at the channel exit.

The target consisted of samarium oxide disks 10 mm in diameter and 1 mm thick placed perpendicularly to the beam with the center on the beam axis. The angular correlation was measured with the aid of two scintillation counters; one was fixed and the other rotated in the vertical plane so that the axis of rotation coincided with the beam axis.

A block diagram of the electronic equipment is given in Fig. 1. A single-channel analyzer passed only pulses corresponding to 440-keV γ rays, so that a 400-channel SA-40 pulse-height analyzer recorded only the spectra in coincidence with these γ rays. The position of the discriminator window is shown in Fig. 2.

The measurements were made at angles $\theta = 90^\circ$,

135° , and 180° . From the spectra we determined the area of the photopeak at 337 keV.

The background was found from the measured shape of the 325-keV γ line of Cr^{51} with allowance for the contribution from random coincidences; the latter was calculated on the basis of the known resolution of the coincidence circuit. At 180° we also took into account the additional background produced by 511-keV γ lines arising in the annihilation of positrons.

The absence of any systematic side effects associated with the experimental arrangement was checked with measurements of the known correlation between the 1.17- and 1.33-MeV γ lines from the Ni^{60} spectrum.

The table lists the experimental and theoretical values of $W(\theta)/W(\pi/2) - 1$ for γ rays of the 440–330 keV cascade [$W(\theta)$ and $W(\pi/2)$ are the numbers of counts at the angles θ and $\pi/2$ respectively; the listed errors are statistical]. The theoretical values were calculated on the basis of the paper of Ferentz and Rosenzweig^[3] with allowance for corrections for the finite solid angles

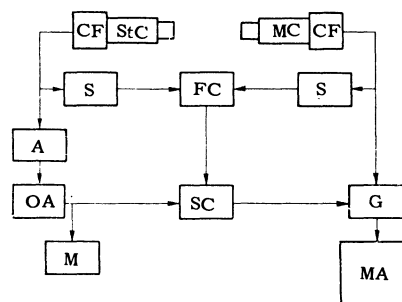


FIG. 1. Block diagram of the experiment: StC – stationary counter, MC – movable counter, CF – cathode follower, S – shaping circuit, FC – fast coincidences ($\tau = 4.8 \times 10^{-8}$ sec), A – linear amplifier, OA – single-channel analyzer, SC – slow coincidences (2×10^{-6} sec), G – gate, MA – 400 channel SA-40 analyzer, M – monitor.

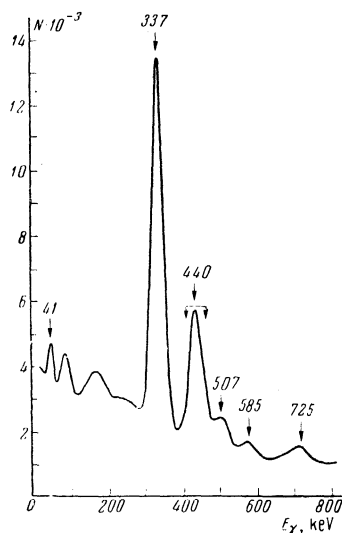


FIG. 2. Gamma spectrum for the capture of thermal neutrons by Sm^{149} . The position of the window of the single-channel analyzer is shown in the figure (N is the number of counts).

subtended by the counters^[4] and for the extended source^[5] with four values of spin: 0, 2, 3, and 4. Here we took the multiplicity of the γ rays to be two, as has been done by Groshev et al.^[6] It follows from the table that the spin of the 777-keV level is 4.

We also made an additional analysis of the experimental results under the assumption that the $E_{II} - E_I$ transition includes an admixture of M1 transitions. For mixed transitions of the type $2[xE2 + (1-x)M1]2(2)0$, where x is the fraction of E2 transitions, no single value of x is compatible with both experimental points. Figure 3 shows the calculated correlation curves when the quantity $2[xE2 + (1-x)M1]2(2)0$ is chosen so that the curves pass through one of the experimental points. Curve b ($x = 0.99$) passes through the experimental point at 180° , but is significantly at variance with experiment at 135° , while curve a ($x = 0.78$) passes through the experimental point at 135° , but strongly disagrees with the experimental point at 180° .

Under the assumption of spin 3, all mixtures with M1 transitions lead to values of $W(\theta)/W(\pi/2)$ less than unity. By way of example, the calculated curve c in Fig. 3 represents a transition of the type $3(0.98E2 + 0.02M1)2(2)0$.

Hence the values of 2 and 3 for the spin of the E_{II} level are also excluded if it is assumed that the transitions are mixed.

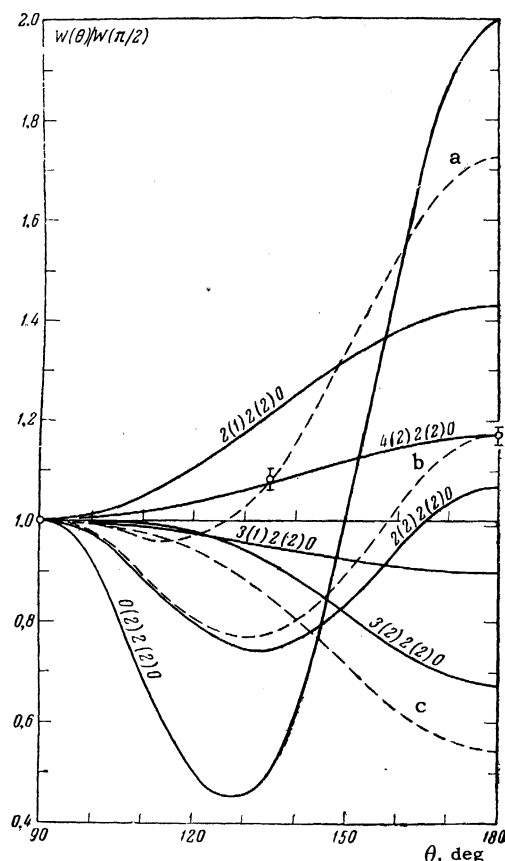


FIG. 3. Correlation curves for various possible values of the spin of the second level and for different multiplicities of the $E_{II} - E_I$ transition; the solid lines represent pure transitions and the dashed lines represent mixed transitions. Curves a, b, and c correspond to transitions of the type $2(0.72E2 + 0.28M1)2(2)0$, $2(0.99E2 + 0.01M1)2(2)0$, and $3(0.98E2 + 0.02M1)2(2)0$, respectively.

The analysis also indicates that the $E_{II} - E_I$ transition is pure. Preliminary estimates show that the effect of γ rays of energy greater than 440 keV on the correlation is not important. A spin of 4 and not 3 is indicated by the fact that the 777-keV level belongs to the vibrational band of the ground state. A spin of 2 is ruled out by the experimental results, since no 777-keV line is visible in the spectrum obtained with one crystal.

In conclusion, we take this opportunity to thank Professor H. Hulubei for his constant interest in the work, E. Iliescu and D. Dragomirescu for aid in constructing the equipment, and A. Bedescu-

θ , deg	$W(\theta)/W(\pi/2) - 1$				
	Experimental value	Theoretical value (corrected)			
		cascade 0(2)2(2)0	cascade 2(2)2(2)0	cascade 3(2)2(2)0	cascade 4(2)2(2)0
135	0.082 ± 0.020	-0.464	-0.261	-0.081	0.068
180	0.170 ± 0.015	0.968	0.076	-0.317	0.158

Singureanu and A. Gelberg for discussions of the results.

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ABSORPTION OF STOPPED π^- -MESONS IN CARBONV. S. DEMIDOV, V. G. KIRILLOV-UGRYUMOV, A. K. PONOSOV, V. P. PROTASOV, and
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Stars produced by π^- mesons stopped in a four-liter propane bubble chamber were investigated. The distribution of the stars with respect to the number of prongs was determined; the mean number of prongs was found to be 0.84. The distribution of two-prong stars with respect to the opening angle between the prongs was also measured and found to have a sharp peak near 180° . Both results indicate a two-nucleon mechanism of the π^- -meson absorption in the carbon nucleus. The probability of π^- -meson absorption by an np pair is 70%.

THE lack of clear-cut results concerning the absorption mechanism of π mesons in complex nuclei has led us to investigate this process for negative π mesons stopped in matter.

For this purpose we analyzed the pictures from a propane bubble chamber taken earlier to study the scattering of slow π^- mesons.^[1] The use of the propane bubble chamber determines automatically the nucleus with which the interaction has occurred, for in propane the stopped π^- mesons are absorbed only by the carbon.

We selected about 3500 π^- mesons stopped inside the chamber, out of which 1130 were used to construct the distribution of π meson stars with respect to the number of prongs, and 1180 to study the distribution with respect to the opening angle between the prongs.

In the first case, the experimental data were corrected for the detection efficiency of a given number of prongs in a star. Such a correction, as was found out, is essential only for single-prong stars and starless stopping events, and is due to the fact that prongs directed at angles close to 0 and 180° with respect to the direction of motion of the primary meson are often omitted in scanning. In order to make the correction, we constructed the distribution of 1000 single-prong stars with respect to the angle between the prong and the primary meson. From general considerations, this distribution should be isotropic, whereas in the experiment dips were observed at angles $0-20^\circ$ and $160-180^\circ$. These dips were considered as being due to the omission of single-prong stars, or rather to accounting them as starless stopping events. The corresponding overshoot amounted to 12.7%.

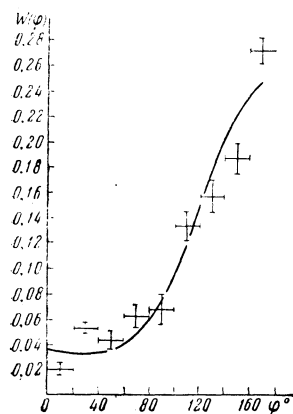
The study of the tracks of secondary particles emitted from π -meson stars leads to the conclusion that the secondary particles are exclusively protons. In this connection it should be noted that the above-mentioned correction does not take into account the fact that protons with energy less than 4–5 MeV (range in propane ≤ 0.5 mm) are in practice undetectable. It was, however, possible to account for this fact in the formulae used. The corrected distribution of stars with respect to the number of prongs is shown in the second column of the table (where the values of the corresponding probability are given).

Assuming ^[2] that π^- mesons are absorbed in the nucleus only by nucleon pairs [(np) or (pp)], and that a different number of prongs in the stars is a result of the cascade scattering of the recoil nucleons on the remaining nucleons in the nucleus, we can calculate the distribution of stars with respect to the number of prongs. This distribution was calculated taking not more than one collision of the recoil nucleons inside the carbon nucleus into account.

A comparison of the experimental data with the calculation by the least-squares method showed that the absorption probability of a π^- meson nucleus

Distribution with respect to
the number of prongs N

N	W_{exp}	W_{calc}
0	0.400 ± 0.019	0.379
1	0.401 ± 0.018	0.440
2	0.159 ± 0.012	0.160
3	0.034 ± 0.005	0.021
4	0.006 ± 0.003	0.000



by a (np) pair amounts to 70–80%, by a (pp) pair to 30–20%, and that the probability of a collision inside the nucleus is 60–80%. These results are similar to the earlier ones^[3,4] (more accurately, we found three close pairs of the probability values, lying within the limits given above, and being solutions of the problem).

The distribution with respect to the number of prongs, calculated assuming a 70% absorption probability by a (np) and a 70% probability of a scattering of the recoil nucleon inside the nucleus, is shown in the third column of the table (W_{calc}). (The probability values used correspond to the deepest minimum of the least-squares method.)

We also constructed the distribution of two-prong stars with respect to the projection of the opening angle between the prongs, and the distribution with respect to the projection of the angle between a prong and the direction of motion of the primary π meson. As expected, the latter distribution was found to be isotropic. The distribution with respect to the projection of the opening angle between the prongs shows a characteristic increase with increasing value of the projection of the angle (see the figure). This fact confirms the assumed nucleon-pair mechanism for the absorption of the π^- meson in the nucleus.

For a qualitative analysis, this distribution has been calculated assuming that the scattering of nucleons inside the nucleus is isotropic in the center-of-mass system of the two nucleons. In

the calculation we used the above-given values of the probabilities of collision and absorption.

The obtained distribution is of the form:

$$W(\varphi) = A \left(\frac{\cos \varphi \sin 2\varphi}{4} + \frac{\sin^3 \varphi}{2} - \frac{\varphi}{2} \cos \varphi \right) + \frac{A}{4} B(\varphi),$$

$$B(\varphi) = \begin{cases} \cos \varphi & 0 \leq \varphi \leq \pi/2, \\ -\cos \varphi & \pi/2 \leq \varphi \leq \pi, \end{cases}$$

where φ is the projection of the opening angle between the prongs. The distribution $W(\varphi)$, normalized to the total number of stars, is shown in the figure by the solid curve. It can be easily seen that the calculated distribution fits well the experimental points.

We can thus claim that the obtained preliminary results on the absorption of stopping π mesons in carbon are not in disagreement with the two-nucleon absorption mechanism, and that the absorption probability by a nucleon pair is apparently independent of the energy in the range up to ~ 200 MeV.^[3,4]

In conclusion we would like to express our gratitude to Prof. A. I. Alikhanyan and to L. P. Kotenko for supplying the film for analysis and for their constant interest in the work and helpful discussion. We thank Z. I. Volobueva for help in the measurements.

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TWO NEW SHORT-LIVED ISOMERS $\text{Ir}^{187\text{m}}$ AND $\text{Ir}^{189\text{m}}$

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Results of investigation of the millisecond isomeric γ activity previously observed^[1] when osmium is bombarded by fast protons are presented. It is found that proton bombardment of an osmium sample with a natural isotopic composition results in the formation of two isomers, $\text{Ir}^{187\text{m}}$ ($T_{1/2} = 29$ msec, $E_\gamma = 115$ keV) and $\text{Ir}^{189\text{m}}$ ($T_{1/2} = 14$ msec, $E_{\gamma 1} = 120$ keV, $E_{\gamma 2} = 180$ keV and $E_{\gamma 3} = 300$ keV). It is shown that isomeric states of these nuclei are produced in the reactions $\text{Os}^{188}(\text{p}, 2\text{n})\text{Ir}^{187\text{m}}$ and $\text{Os}^{190}(\text{p}, 2\text{n})\text{Ir}^{189\text{m}}$.

MOROZOV, Yampol'skiĭ, and one of the authors^[1] reported that the bombardment of osmium of natural isotopic composition by 20-MeV protons leads to an intensive emission of γ rays with a half-life $T_{1/2} = 10^{-2}$ sec and γ -ray energy $E_\gamma = 0.320$ MeV. Using enriched samples of osmium isotopes, the authors of the present paper were able to identify the observed isomeric radiation.

For the experiment, we used 20-MeV protons from a linear accelerator. The experimental arrangement and technique were similar to that described earlier.^[2]

We had at our disposal samples of osmium enriched in the isotopes Os^{188} , Os^{189} , Os^{190} , and Os^{192} . The abundance of the various isotopes in these samples is given in the table.

Figure 1 shows three spectra of isomeric γ radiation. One of them corresponds to radiation produced as a result of the bombardment of an osmium sample of natural isotopic composition by protons, the other two represent samples enriched in Os^{188} and Os^{190} . Comparison of the γ -ray yields of different energies arising from the proton bombardment of targets enriched in different osmium isotopes showed that two new isomers are produced. An isomer with a γ -transition

energy of 115 keV is produced from Os^{188} isotopes and an isomer with γ -transition energies of 120, 180, and 300 keV is produced from Os^{190} . The radiation spectrum of both isomers has an intense peak of 650-keV x rays. The half-lives of these isomers proved to be 29 ± 2 msec for the first and 14 ± 1 msec for the second. The decay curves of the isomeric activity are shown in Fig. 2.

To identify the reactions leading to the production of the observed isomers we determined the excitation functions of the corresponding isomeric activities (Fig. 3). The excitation function of the 14-msec activity was determined from the photopeak of the 300-keV γ transition with a target of natural osmium (~ 20 mg/cm² thick); the excitation function of the 29-msec activity was determined from the x-ray peak with a target enriched in Os^{188} (~ 60 mg/cm² thick). The obtained value of the threshold and the shape of the excitation function support the conclusion that the isomeric states in both cases are produced in a (p, 2n) reaction. Hence the observed γ activities correspond to the decay of two new millisecond isomers $\text{Ir}^{187\text{m}}$ and $\text{Ir}^{189\text{m}}$, which are products of the nuclear reactions

Sample	Isotope abundance, %						
	Os^{184}	Os^{186}	Os^{187}	Os^{188}	Os^{189}	Os^{190}	Os^{192}
Natural Os	0.018	1.582	1.64	13.27	16.14	26.38	40.97
Os^{188}	—	—	—	75.4	11.7	8.7	3.9
Os^{189}	—	—	2.4	6.4	71.9	19.3	—
Os^{190}	<0.1	<0.1	0.3	1.8	4.4	76.1	17.4
Os^{192}	—	—	—	—	—	1.9	98.1



The Ir^{187m} isomer decays with a half-life of 29 ± 2 msec. Its γ spectrum consists of a peak corresponding to a (115 ± 5) -keV transition and an intense peak of iridium x rays. From the ratio of the areas of these two peaks we estimated the conversion coefficient α_K of the 115-keV transition. The obtained value $\alpha_K = 7 \pm 2$ was compared with the calculated values:^[3] $\alpha_K(E1) = 0.23$; $\alpha_K(E2) = 0.59$; $\alpha_K(E3) = 1.17$; $\alpha_K(E4) = 2.5$; $\alpha_K(E5) = 5.5$; $\alpha_K(M1) = 3.5$; and $\alpha_K(M2) = 22.4$. It is seen that the 115-keV radiation most probably should be attributed to a mixture of transitions $E1 + M2$ or $M2 + E3$. The first conclusion on the multipolarity of the 115-keV transition is in poor agreement with the lifetime of the Ir^{187m} isomeric state. It is also quite possible that the decay scheme includes a transition preceding this one. Such a transition may not have been noticed if its energy is less than 30 keV or if it is strongly converted.

Fettweis and Campbell^[4] recently reported that they obtained a new isomer Ir^{194m} from the

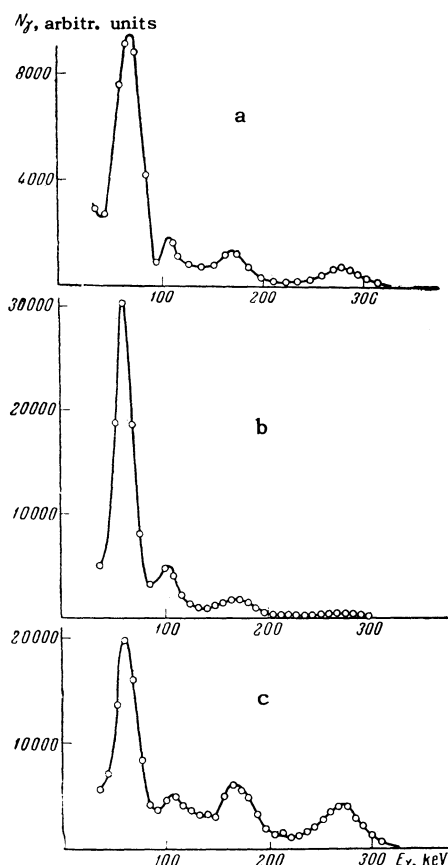


FIG. 1. Spectra of isomeric γ radiation produced in the bombardment of osmium samples by protons; a) natural isotopic composition, b) enriched in Os^{188} , c) enriched in Os^{190} .

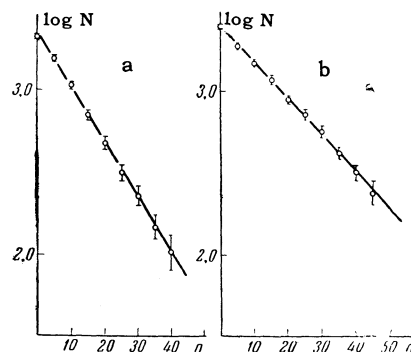


FIG. 2. Decay of isomeric activities for Os^{190} (a) and Os^{188} (b) (N is the number of recorded pulses, n is the channel number of the time analyzer). The channel widths are: a) 1.5 msec, (b) 2.1 msec. The measurements were made from the γ -ray photopeak at 0.3 MeV (a) and from the x-ray peak (b); the constant background has been subtracted in both cases.

$\text{Ir}^{193}(n, \gamma)\text{Ir}^{194m}$ reaction. The decay characteristics of this isomer ($T_{1/2} = 32$ msec, $E_\gamma = 115$ keV, $\alpha_K \approx 6$) agree, within the limits of experimental accuracy, with the values obtained by us for Ir^{187m} . Despite the good agreement of the characteristics of the isomeric radiation, we can rule out the possibility that the same isomer was observed in both experiments, since in^[4] the isomer Ir^{187m} could not, in principle, have been obtained, while in our case we could not have obtained the isomer Ir^{194m} .

The Ir^{189m} isomer decays with a half-life of 14 ± 1 msec. Its radiation spectrum contains an x-ray and three peaks corresponding to γ transitions of 120 ± 5 , 180 ± 10 , and 300 ± 10 keV. The existence of excited states of energy 113 and 305 keV in Ir^{189} has been established by Kryukova et al.^[5]

According to the data of^[5], the Ir^{189} state with an excitation energy of 305 keV is not isomeric. It can therefore be assumed that the isomeric level is higher and has an excitation energy ~ 350 keV. The transition from the isomeric level may not have been detected in our experiment if it is strongly converted or located close to the energy of the x-ray peak. At present, the isotopes Ir^{191} and Ir^{193} are known to have isomeric states with spin and parity $11/2^-$.^[6] If it is assumed that the shift in this level in the case of Ir^{189} follows the same pattern as in the case of Ir^{191} and Ir^{193} (see Fig. 4), then the obtained decay scheme for the isomeric state of Ir^{189} is in good agreement with the experimental results.¹⁾ We then obtain an explanation of the sharp decrease in the lifetimes of isomeric states of these nuclei with decreasing

¹⁾See note added in proof.

FIG. 3. Excitation functions for isomeric activities in the $\text{Os}^{190}(\text{p}, 2\text{n})\text{Ir}^{189\text{m}}$ and $\text{Os}^{188}(\text{p}, 2\text{n})\text{Ir}^{187\text{m}}$ reactions (a and b, respectively).

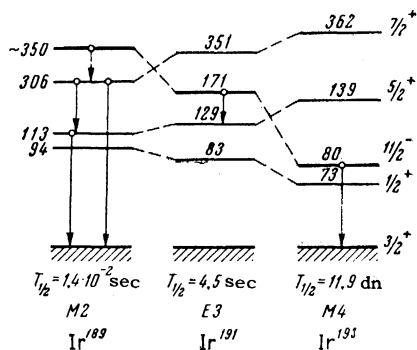
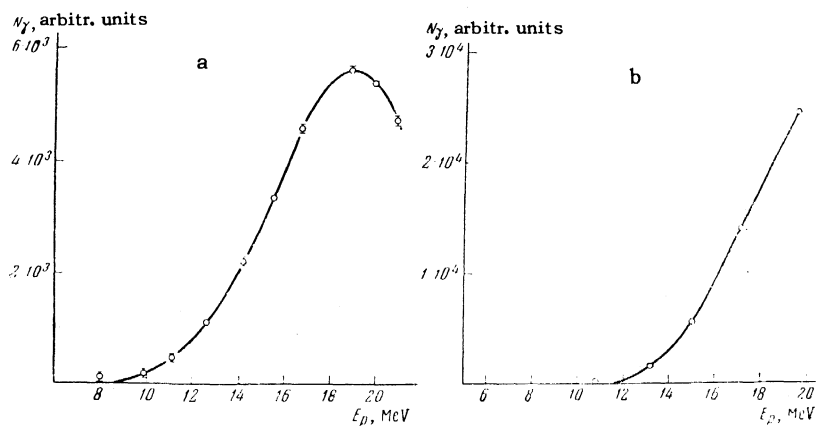


FIG. 4. Succession of levels of Ir^{189} , Ir^{191} , Ir^{193} and the suggested decay scheme for $\text{Ir}^{189\text{m}}$. The numbers denote the energy of the levels (in keV); the half-life of the isomeric state $11/2^-$ and the multipolarity of the transition from this level are shown for the respective schemes.

number of neutrons, so that the multipolarity of the isomeric transitions decreases.

The authors thank A. P. Klyucharev for great assistance in carrying out this experiment and L. K. Peker for helpful discussions of the results. We are indebted to V. S. Zolotarev and V. M. Gusev for supplying the osmium isotopes.

Note added in proof (March 4, 1963): The authors have made a more accurate determination of the energies of the γ transitions in $\text{Ir}^{189\text{m}}$: 114 ± 5 , 184 ± 5 , and 297 ± 5 keV. The following levels for Ir^{189} have recently been established:^[7] 113.75 ($5/2^+$), 300.5 ($7/2^+$), and 372.1 ($11/2^-$). The energies of the corresponding transitions — 113.75, 186.75, and 300.5 keV — are in good agreement with our values. The transition from the 372.1-keV level is practically fully converted. Hence the decay scheme of $\text{Ir}^{189\text{m}}$ in Fig. 4 should be regarded as confirmed; the state with an energy 372.1 keV ($11/2^-$) is isomeric.

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TIME VARIATION OF THE SPECTRAL COMPOSITION OF THE RADIATION FROM AN OPTICAL MASER OF Nd-ACTIVATED GLASS

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It is shown that the number of spectral lines produced during a flash of an optical maser of Nd-activated glass varies with time as do their wavelengths.

SNITZER^[1] has reported that the radiation of an optical maser of Nd-activated glass consists of several lines located in the region $1.061-1.062 \mu$.

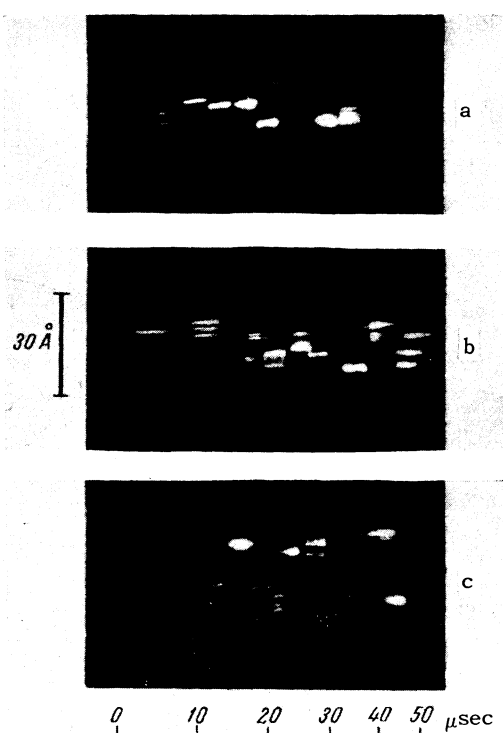
The present paper presents data on the change in time of the spectral composition of this maser. The maser was a cylinder of optical glass containing neodymium, of length 60 mm and diameter 8 mm. The concentration of Nd_2O_3 in the glass was 2%. The spectral analysis was accomplished with a diffraction spectrograph with a dispersion of $14 \text{ \AA/mm}$, the time analysis with an image converter. The scanning was done in a direction perpendicular to the direction of dispersion of the spectrograph.

The figure shows time-scan photographs of the spectrum of the radiation of the Nd-glass maser for different levels of the "pump" above the generation threshold. The spectral lines appear as horizontal lines the direction of which is the same as the direction of the time scan and the length of which is determined by the height of the spectrograph slit. For our choice of slit height, the time resolution of the record is about $5 \mu\text{sec}$, although the registering apparatus has a resolution of about $2 \times 10^{-7} \text{ sec}$.

From the photographs it can be seen that the spectral composition of the separate bursts of radiation making up the generated flash changes from burst to burst. The spectrum of the first burst consists, as a rule, of a single spectral line. In later bursts there occurs a change in wavelength of the generated light and the appearance of several lines is possible. As the intensity of the "pump" is increased there is a tendency for the number of lines generated in each burst to increase, and the number of lines in one burst and the spectral region occupied by them increase toward the end of the flash. The distance between the extreme lines in one burst can reach $20 \text{ \AA}$. The wavelength interval encompassing the generated lines belonging to all the bursts widens as the "pump" power

is increased, and when this exceeds the threshold by 70%, the interval amounts to $30 \text{ \AA}$ (see the figure, b).

A consideration of these photographs leads to certain conclusions about the processes occurring in this maser. The first burst of radiation always corresponds to the transition between that pair of sublevels of the terms $^4F_{3/2}$ and $^4I_{11/2}$ which have the most favorable parameters for stimulated emission. The radiation in the second burst comes from another sublevel of the $^4F_{3/2}$ term, since the one from which the transitions occurred in the first burst has become empty while the pop-



Time scan of the spectrum of radiation from a Nd-glass optical maser for pump levels in excess of threshold by: a - 20%, b - 40%, c - 70%.

ulation of the other sublevels of this term has continued to grow on account of the "pump," and the sublevel of the $^4I_{11/2}$ to which transitions were made in the first burst is now inverted with respect to the other sublevels of this term.

The generation of several lines in a single burst can be explained in the following way. When the "pump" energy is far in excess of threshold, the required inversion turns out to be attained for several pairs of sublevels at once. At the initial

moment of emergence of the burst the resonator still has a wide spectrum and therefore transitions from several levels are initiated simultaneously; thus several lines are generated in a single burst.

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PHOTODISINTEGRATION OF Li^7

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The energy spectra and angular distributions of protons and tritons produced in the photodisintegration of Li^7 are investigated. In the energy spectrum of phototritons produced at $E_{\gamma\text{max}} = 30$ MeV the observed groups correspond to excitation energies of 14.1, 16.2, 18.0, 19.6, 21.5, 23.5, and 25.3 MeV. In most of the energy ranges the triton angular distributions have the form $\sim \sin^2 \vartheta$. Pronounced peaks have also been found in the Li^7 photoproton spectra. However, for $E_{\gamma\text{max}} = 25$ and 30 MeV the bulk of the observed protons are associated with transitions to excited states of the He^6 nucleus.

INTRODUCTION

It has been established that the probability of the reaction $\text{Li}^7(\gamma, t)\text{He}^4$ is comparable to that of the (γ, n) and (γ, p) reactions in the photodisintegration of Li^7 . The energy spectra of the tritons have revealed sharply defined groups, leading to the hypothesis that at relatively low excitation energies the reaction occurs via a compound nucleus^[1,2]. The angular distributions, which are close to $1 + \sin^2 \vartheta$ for the principal triton groups, represent dipole transitions to excited Li states with spin $5/2$.

For $\text{Li}^7(\gamma, t)\text{He}^4$, Czyz^[3] has considered theoretically the direct photodisintegration mechanism of Li^7 as a system consisting of an α particle and

a triton in the $P_{3/2}$ state. In this model, the triton angular distributions should have the form $1 + 1.5 \sin^2 \vartheta$.

In^[4] we obtained an angular distribution of 7.5–15-MeV tritons represented by $a + b \sin^2 \vartheta$ with $b/a = 2.5$. It is difficult to reconcile this distribution with the aforementioned hypotheses regarding the mechanism of $\text{Li}^7(\gamma, t)\text{He}^4$.

Shardonov and Shevchenko^[5] have obtained the energy spectrum and angular distribution of protons from the photodisintegration of Li^7 ($E_{\gamma\text{max}} = 17.5$ MeV). The energy groups in the proton spectrum were regarded as corresponding to excited Li^7 levels.

In the present work we have investigated the energy spectra and angular distributions of protons and tritons produced in the photodisintegration of Li^7 at different values of $E_{\gamma\text{max}}$.

EXPERIMENTAL PROCEDURE

The experimental technique and method of identifying and measuring the energies of charged particles produced in photodisintegration are similar to those described in^[4] except for some changes in the apparatus. A lithium target 8 mg/cm² thick was placed in a vacuum chamber in conjunction with two identical counter telescopes. The head counters of the telescopes, giving pulses proportional to the energy loss $\Delta E \sim dE/dx$, were proportional counters filled with argon and a small admixture of methane; this made possible registration of photoreaction products having somewhat lower energies, and also enhanced the energy (ΔE) resolution of the telescopes.

In order to reduce the distortions of pulses

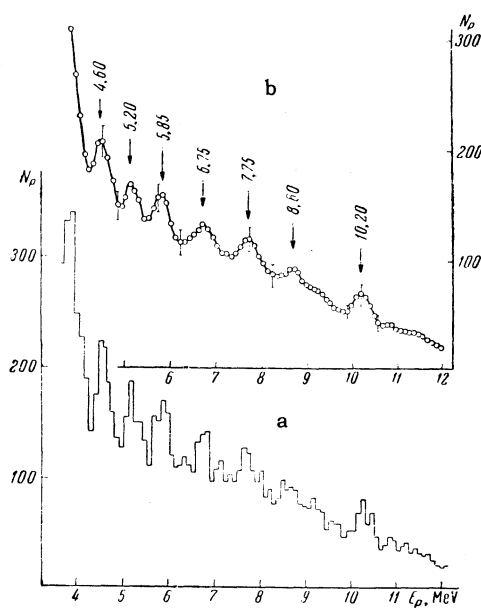


FIG. 1

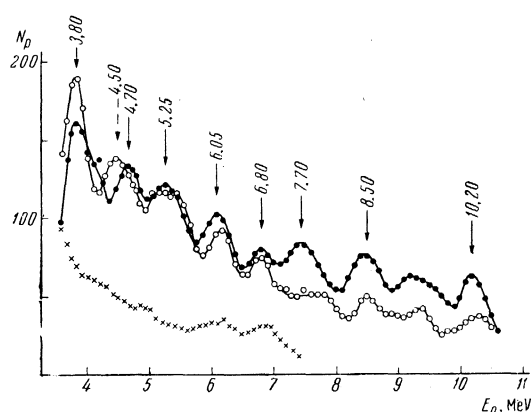


FIG. 2. Smoothed energy spectra of photoprotons from Li^7 for $E_{\gamma\text{max}} = 20$ MeV ($\times$), 25 MeV ($\circ$), and 30 MeV ($\bullet$). Ordinates represent the proton yield (in arbitrary units) relative to an identical total γ -ray energy. The arrows indicate the spectral peaks.

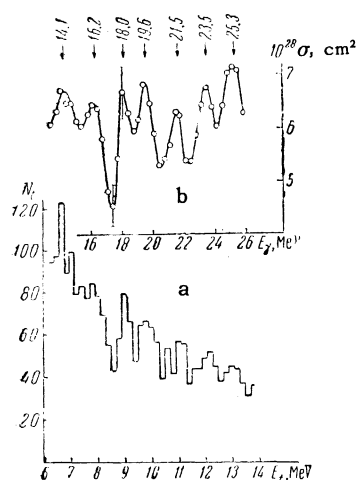


FIG. 3

representing reaction products resulting from the integration of many small background pulses in the amplification channels, the pulses from the photomultipliers of scintillation counters in the tail ends of the telescopes (pulses proportional to the particle energy E) were sent through "fast" integral discriminators. This improved the energy (E) resolution of the telescopes.

In these experiments we did not employ special circuits to stabilize the amplification of the scintillation counters.

EXPERIMENTAL RESULTS

Figure 1a shows the combined energy spectrum of protons emitted in the photodisintegration of Li^7 by bremsstrahlung having $E_{\gamma\text{max}} = 30$ MeV at the angles 54° , 72° , 90° , 108° , and 126° . In Fig. 1b the same spectrum has been smoothed out; the arrows point to peaks appearing in all the proton spectra

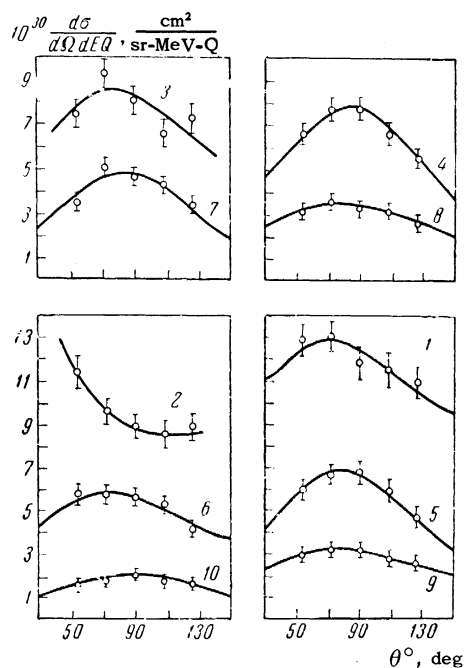


FIG. 4. Angular distributions of protons from Li^7 photo-disintegration at $E_{\gamma\text{max}} = 30$ MeV for different energy intervals: 1—3.8–4.2 MeV; 2—4.5–4.9 MeV; 3—5.2–5.7 MeV; 4—5.8–6.4 MeV; 5—6.6–7.2 MeV; 6—7.7–8.3 MeV; 7—8.6–9.2 MeV; 8—9.4–10.8 MeV; 9—10.2–10.8 MeV; 10—11.0–12.1 MeV. The smooth curves were obtained by least squares using the formula $a + b \sin^2 \theta (1 + \gamma \cos \theta)^2$. Ordinates represent the differential cross section per effective quantum.

registered by both telescopes at the indicated angles.

Figure 2 shows the smoothed energy distributions of protons emitted in the photodisintegration of Li^7 for $E_{\gamma\text{max}} = 20$, 25, and 30 MeV ($\vartheta = 90^\circ$).

Figure 3a shows the combined energy spectrum of tritons measured under the same conditions as the proton spectrum in Fig. 1. Fig. 3b shows the cross section curve for $\text{Li}^7(\gamma, t)\text{He}^4$ derived from the smoothed triton spectrum. The arrows indicate the peaks appearing in all measurements with both telescopes at the given angles.

The angular distributions of photoprotons and phototritons for different energy intervals are shown in Figs. 4 and 5. The angular distributions were approximated by an analytic expression of the form $a + b \sin^2 \vartheta (1 + \gamma \cos \vartheta)^2$. The values of a/b and γ determined by least squares, for the proton and triton angular distributions in the separate energy intervals, are given in Tables I and II.

Table III gives the cross sections for the production of protons, deuterons, and tritons per effective quantum in the interval 4.8–7.8 MeV for different values of $E_{\gamma\text{max}}$ ($\vartheta = 90^\circ$). The data in Table III include only the statistical errors and the errors associated with inaccurate mass resolution of the particles in treating the data.

Table I

Proton energy interval, MeV	Proton angular distribution $a + b \sin^2 \theta (1 + \gamma \cos \theta)^2$		Proton energy interval, MeV	Proton angular distribution $a + b \sin^2 \theta (1 + \gamma \cos \theta)^2$	
	a/b	γ		a/b	γ
3.8-4.2	2.8 ± 0.5	0.4 ± 0.1	7.7-8.3	1.43 ± 0.30	0.35 ± 0.1
4.5-4.9	-5.0 ± 0.7	—	8.6-9.2	0.28 ± 0.04	0.08 ± 0.1
5.2-5.7	3.2 ± 0.7	0.25 ± 0.1	9.4-10.0	1.35 ± 0.25	0.24 ± 0.1
5.8-6.4	0.57 ± 0.07	0.135 ± 0.1	10.2-10.8	1.28 ± 0.25	0.23 ± 0.1
6.6-7.2	0.61 ± 0.08	0.20 ± 0.1	11.0-12.1	0.69 ± 0.15	0.0 ± 0.1

Table II

Triton energy interval, MeV	Triton angular distribution $a + b \sin^2 \theta (1 + \gamma \cos \theta)^2$	
	a/b	γ
6.7-7.2	-0.07 ± 0.12	-0.05 ± 0.05
7.4-8.4	0.15 ± 0.15	-0.2 ± 0.1
8.6-11.4	0.025 ± 0.08	-0.05 ± 0.05
11.8-12.6	0.50 ± 0.15	-0.3 ± 0.1
6.7-12.6	0.1 ± 0.08	-0.12 ± 0.1
7.5-15 [4]	0.4 ± 0.07	—

Table III

Particle	$10^{30} \frac{d\sigma}{d\Omega dE dQ} \cdot \frac{\text{cm}^2}{\text{sr-MeV-Q}}$	
	$E_{\gamma \text{ max}} = 20 \text{ MeV}$	$E_{\gamma \text{ max}} = 25 \text{ MeV}$
p	2.7 ± 0.2	8.6 ± 0.4
d	0.11 ± 0.03	0.40 ± 0.12
t	1.36 ± 0.24	1.36 ± 0.12

Table IV

E_t , MeV	E_{γ} , MeV	Energy level of Li^7 [6]	E_t , MeV	E_{γ} , MeV	Energy level of Li^7 [6]
6.6 ± 0.2	14.1 ± 0.3	14.0	10.9 ± 0.2	$(21.5 \pm 0.3)^*$	(21.5)
7.8 ± 0.2	16.2 ± 0.3	16.2	12.1 ± 0.2	(23.5 ± 0.3)	(23.5)
9.0 ± 0.2	18.0 ± 0.3	17.5	13.0 ± 0.2	(25.3 ± 0.3)	—
9.8 ± 0.2	19.6 ± 0.3	—			

*Less reliable values are given in parentheses.

DISCUSSION OF RESULTS

The peaks of the cross section for $\text{Li}^7(\gamma, t)\text{He}^4$, denoted by arrows in Fig. 3, were compared with the excited levels of Li^7 in [6]. Table IV shows that there is good agreement for the levels at 14.1, 16.2, 21.5, and 23.5 MeV. This can indicate that the reaction $\text{Li}^7(\gamma, t)\text{He}^4$ involves the formation of a compound nucleus. Levels corresponding to the peaks at 19.6 and 25.3 MeV still apparently have not been observed.

The angular distributions of tritons from $\text{Li}^7(\gamma, t)\text{He}^4$, calculated from the theory of resonance γ -ray absorption with compound-nucleus formation, are given in [1,2]. None of these distributions agrees with the angular distributions of the form $\sim \sin^2 \theta$ in the present work (Table II). Tritons associated with transitions to excited Li^7 states having a specified spin J can be emitted with different orbital momenta l ; therefore the angular distributions should have the form [7]

$$d\sigma(\theta)/d\Omega \sim \sum_{l'l''} \text{Re}\{R_l^* R_{l''}\} Q_{l''}^l P_L(\cos \theta)$$

(R_l^J , and $R_{l''}^J$ are the matrix elements of the operator $\hat{R} = i\hat{S} - \hat{1}$, where $\hat{S}$ is the scattering matrix, $\hat{1}$ is the unit matrix, $Q_{l''}^l$ are numerical coefficients

depending on l and L , and P_L are Legendre polynomials), if we do not consider the possible interference of the absorption of γ quanta of different multipolarities. At the present time it is impossible to calculate theoretically the matrix elements R_l^J determined by the nuclear structure. We cannot exclude the possibility that in the reaction under consideration the relations between the matrix elements will lead to the vanishing of the isotropic part of the angular distribution. However,

FIG. 5. Angular distributions

of phototritons from $\text{Li}^7(\gamma, t)\text{He}^4$. Ordinates represent the reaction cross section in arbitrary units at $E_{\gamma \text{ max}} = 30 \text{ MeV}$ in different energy intervals: a-6.7-7.2 MeV; b-7.4-8.4 MeV; c-8.6-11.4 MeV; d-11.8-12.6 MeV. The continuous curves for the first energy interval were obtained using the formulas $\sin^2 \theta$ and $1 + \sin^2 \theta$; for the other intervals the curves were obtained by least squares from the formula $a + b \sin^2 \theta (1 + \gamma \cos \theta)^2$.

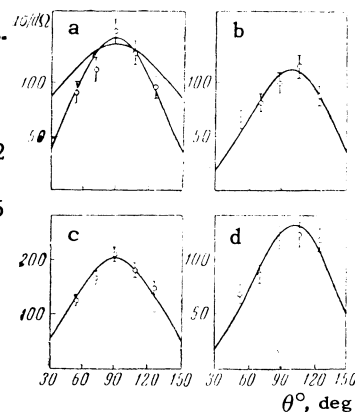


Table V

Proton group energy, MeV	% of protons accompanying transitions to second and higher excited states of He^6		Proton group energy, MeV	% of protons accompanying transitions to second and higher excited states of He^6	
	$E_{\gamma\max} = 25 \text{ MeV}$	$E_{\gamma\max} = 30 \text{ MeV}$		$E_{\gamma\max} = 25 \text{ MeV}$	$E_{\gamma\max} = 30 \text{ MeV}$
3.8	50*	50*	6.80	28	40
4.6	90	90	7.70	30	50
5.25	75	85	8.60	45	85
5.85	34	60	10.20	0	45

*% of protons accompanying transitions to the third and higher excited states of He^6 .

it may appear surprising that this shape of the angular distribution is observed for a broad energy interval containing several excited levels.

In [3] the total integral and differential cross sections for $\text{Li}^7(\gamma, t)\text{He}^4$ were calculated on the two-particle model of Li^7 , in which it is assumed that Li^7 consists of an α particle and a triton in the $P_{3/2}$ state. The general formula representing the differential cross section for Li^7 disintegration into an α particle and a triton through the dipole absorption of γ quanta is

$$\sigma(\vartheta) = \frac{\sigma_{PD}}{100\pi} \left(\frac{25}{2} + \frac{75}{4} \sin^2 \vartheta \right) + \frac{1}{4\pi} \sigma_{PS} \mp \frac{1}{4\pi} \sqrt{2\sigma_{PS}\sigma_{PD}} \cos(\delta_2 - \delta_0) \left(\frac{3}{2} \sin^2 \vartheta - 1 \right);$$

σ_{PD} and σ_{PS} were calculated assuming square-well potentials of different depths for states with different orbital momenta (V_S , V_P , and V_D for S, P, and D states, respectively). V_S and V_D were selected for agreement with the experimental angular distributions of the form $1 + 1.5 \sin^2 \vartheta$ given in [1] for 4.7-MeV γ rays. On this model V_S and V_D could probably be selected to make the isotropic part of the angular distribution so small as to produce agreement with the results of the present work. It must be remembered, however, that the calculations represent a very rough approximation.

In order to account for the experimental angular distributions of phototritons and the structure of the spectra it must evidently be assumed that both of the discussed processes of phototriton production actually take place.

The identification of groups in the photoproton spectra is hindered by possible transitions to different excited states of the final He^6 nucleus (1.71, 3.4, 6.0, and 9.3 MeV).^[6] The photoproton spectra obtained for three values of $E_{\gamma\max}$ (20, 25, and 30 MeV) permit only tentative conclusions regarding the probabilities of transitions to different excited states for separate groups.

Table V gives the probabilities of transitions to

the second and higher excited states of He^6 , for the different proton groups, with $E_{\gamma\max} = 25$ and 30 MeV. These data suggest that the bulk of the protons produced through photodisintegration by the given bremsstrahlung spectra are associated with transitions leaving the residual nucleus in an excited state.

The angular distributions of photoprotons in the separate groups differ strongly from each other with respect to the ratio of the coefficients a and b (see Table I), and for most of the groups are not in disagreement with the distributions based on the compound-nucleus model. However, the existence of angular distributions having the form $1 + 1.5 \sin^2 \vartheta$ for three-proton energy intervals indicates that direct photoproton emission is also possible.

In conclusion the authors wish to thank G. M. Shklyarevskii for discussions of the results, and also the cyclotron crew.

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NEUTRON POLARIZATION IN THE $D(d, n)He^3$ REACTION

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Values of the polarization of neutrons emitted at various angles θ_n from a deuterium target for deuteron energies $E_d = 12.0 \pm 0.6$ MeV and $E_d = 9.0 \pm 0.7$ MeV have been obtained with the aid of a helium analyzer and by employing the Seagrave phase shifts for $n\alpha$ scattering.

MEASUREMENTS of the polarization of neutrons from the $D(d, n)He^3$ reaction have been carried out by many authors^[1-6] using various methods. In their work discrepancies are observed in the behavior of the maximum value of the neutron polarization P_n as a function of the deuteron energy E_d (see the figure). The angle dependence of the polarization has been measured only up to a deuteron energy $E_d = 8.9$ MeV. It would be interesting to supplement the available experimental data on the polarization of neutrons in the $D(d, n)He^3$ reaction for deuteron energies $E_d = 9-12$ MeV.

The measurements of the neutron polarization were carried out with the extracted beam of the cyclotron of the Institute of Theoretical and Experimental Physics with 12.3 ± 0.3 MeV deuterons. The azimuthal asymmetry of the neutron scattering was measured with a helium analyzer.^[7] A deuterium-saturated 19-mg/cm² thick zirconium target was used.

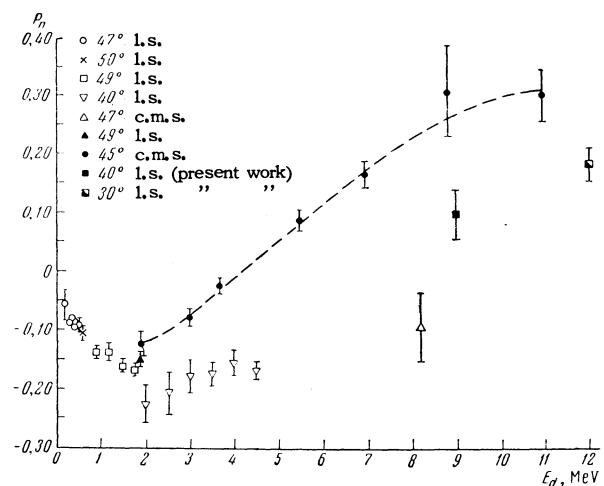
The following values of the polarization of neutrons emitted at various angles θ_n from the deuterium target were obtained for two deuteron energies (Seagrave's $n\alpha$ -scattering phases were used^[8]):

$E_d = 12.0 \pm 0.6$ MeV				
θ_n (l.s.), deg:	20	30	40	50
E_n , MeV:	14.0	13.1	11.9	10.5
P_n , %:	2.2 ± 1.1	18.5 ± 2.3	10.8 ± 3.3	-2.2 ± 3.7
$E_d = 9.0 \pm 0.7$ MeV				
θ_n (l.s.), deg:	20	30	40	
E_n , MeV:	11.4	10.7	9.8	
P_n , %:	-1.6 ± 1.0	1.6 ± 3.4	10.1 ± 3.9	

The positive direction was taken throughout to be along the normal $\mathbf{k}_d \times \mathbf{k}_n$.

The maximum values of the neutron polarization obtained from our data are indicated in the figure.

The authors express their gratitude to the cy-



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DOUBLE MOTT SCATTERING OF ELECTRONS

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The asymmetry of the double scattering of electrons on gold is measured at energies 45–245 keV and angles 90–150°. The data are compared with other experimental and theoretical results.

1. INTRODUCTION

MOTT^[1] was the first to show that electrons scattered by the electric field of a nucleus are partially polarized perpendicular to the plane of scattering. Sherman^[2,3] has calculated accurately the degree of polarization S for scattering by the Coulomb field of a point nucleus at practically all angles and energies. S increases with the nuclear charge; scattering by gold nuclei is of greatest experimental interest. In this case for not very high energies the screening effect of atomic electrons must be taken into account. Calculations in^[4,5] have shown that the degree of polarization can be changed by 10–20% as a result of screening in the energy range 30–120 keV; however, the accuracy of these calculations is evidently not greater than 10%.

The results obtained in^[2-5] indicate that the general behavior of polarization is as follows. In the energy range from a few tens to a few hundreds of keV the magnitude of S does not change by more than 30–40%. S as a function of the scattering angle θ exhibits a broad maximum in the region 120–135°, where polarization reaches ~40% in the case of scattering on gold. The polarization is extremely small at angles 70–160°.

Mott^[1] proposed that the asymmetry of double scattering be observed in order to detect electron polarization. The intensity of doubly scattered electrons is proportional to $1 + S_1 S_2 \cos \varphi$, where φ is the angle between the first and second scattering planes. In the appropriate experimental work one ordinarily observes the ratio x of intensities in the directions $\varphi = 0$ and π , obtaining $S_1 S_2 = (x - 1)/(x + 1)$. Attempts to detect the asymmetry of double scattering did not yield positive results for a long time; this was caused by the lack of sufficiently perfected observational techniques and by the underestimation of the plural and

multiple scattering effects, which mask polarization. Following a successful experiment by Shull et al^[6] a number of double-scattering experiments were performed. However, the results of these investigations did not yield a sufficiently accurate and complete picture of the asymmetry over broad ranges of angles and energies. The results of this former work are included in the discussion of our present results.

Mott scattering constitutes the most sensitive method for determining the degree of polarization $\langle \sigma \rangle$ of an electron beam. The measured quantity is the product $\langle \sigma \rangle S$. Therefore after parity nonconservation in weak interactions was discovered in connection with the measurement of the longitudinal polarization of β electrons, the exact determination of S became urgent. Spivak et al^[7] have reported measurements of S at 120° in the range 45–245 keV. The present work continues that of^[7] by measuring the angular dependence of S for 90–150° at the energies 63, 133, 170, and 245 keV.

2. DESCRIPTION OF APPARATUS

A focused beam 1 of accelerated electrons was directed from a magnetic monochromator to a chamber containing the first scatterer (Fig. 1). The accelerating voltage was calibrated by means of the 187-keV conversion line of In^{114} , and was known to within at least 2% within the entire interval. The position and shape of the beam were checked by means of a fluorescent screen held in the transparent flanged holder 6. The intersection of the beam with the plane of the first scatterer was a circle of about 15-mm diameter. The electron current entering the chamber reached a few tens of microamperes. Following the first scattering at the angle $\theta_1 = 120^\circ$, the electrons passed through a system of diaphragms before impinging

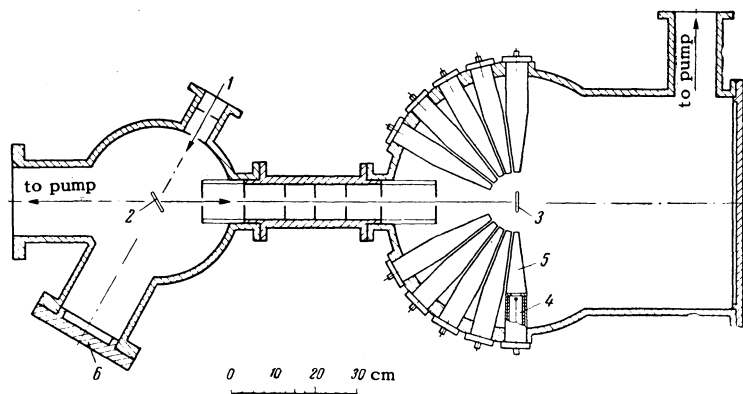


FIG. 1. Schematic drawing of apparatus. 1—electron beam, 2—first scatterer, 3—second scatterer, 4—counter, 5—Plexiglas collimator, 6—flanged holder of fluorescent screen.

on the second scatterer 3 located in a separate chamber at about the distance 70 cm from the first scatterer. Electrons scattered for the second time at angles $\theta_2 = 90^\circ, 105^\circ, 120^\circ, 135^\circ$, and 150° in the directions $\varphi = 0$ and π were registered by the Geiger end-window counters 4.

The scatterers positioned in the first chamber were deposited on aluminum foil $20\text{--}25\ \mu\text{g}/\text{cm}^2$ thick; several scatterers could be interchanged without disturbing the vacuum. The scatterers 3 were deposited on collodion film $10\text{--}50\ \mu\text{g}/\text{cm}^2$ thick stretched on rings 150 mm in diameter. These scatterers, which were interchanged through a vacuum lock, could be rotated around the vertical axis for measurements at $\theta_2 = 90\text{--}120^\circ$.

3. BACKGROUND

One of the most serious difficulties in double-scattering experiments is encountered in eliminating the background of electrons scattered from the walls of the apparatus. The role of this effect was reduced by using large chambers; deep "traps" were installed at the places subjected to the most intense electron bombardment; all walls were lined with Plexiglas and the solid angle subtended by the walls at the counters was reduced as much as possible. Nevertheless, measurements performed with a magnetic spectrometer substituted for one counter revealed that the principal peak of doubly-scattered electrons was accompanied by low-energy electrons. It was shown experimentally that the latter electrons were mainly scattered by the walls of the first chamber and joined the beam entering the second chamber as a result of scattering in the first scatterer. Because of the strong energy dependence of scattering their relative intensity increased sharply following the second scattering. Thus, while they constituted only 2–3% of the beam entering the second scatterer for the accelerating voltage 245 keV, the intensity of "soft" electrons somewhat exceeded that of the main group in the spectrum at the counter window. With

the reduction of the accelerating voltage the number of soft electrons decreases rapidly, becoming negligibly small at 60 keV. (The spectrometer threshold in these measurements was 15 keV.) The low-energy electrons form a quite broad peak; their energy is 4–6 times lower than that of the main electrons, and the two groups are separated by a large energy interval containing practically no electrons. The extraneous electrons could therefore be absorbed completely by filters placed in front of the counter windows. At 63 and 113 keV the filter was a polyethylene film ($4.6\ \text{mg}/\text{cm}^2$) separating the working volume from the vacuum. At 170 and 245 keV additional aluminum filters 10 and $17\ \text{mg}/\text{cm}^2$ thick, were used.

From a careful investigation of scattering conditions in the second chamber we learned the following. For counters at $90\text{--}120^\circ$ scattering from the walls could be neglected entirely. Counters at 135° and 150° were in a less favorable position because of the diminishing cross section as the scattering angle increased and because they could be reached with higher probability by electrons coming from the most intensely irradiated parts of the chamber. Under the actual operating conditions at these angles the extraneous electrons could be reduced to 1 and 2%, respectively. Further experimentation showed that there was no appreciable change in these percentages when the conditions of wall bombardment were changed as the scatterer was installed in its working position. The background could therefore be measured by removing the scatterer from the path of the beam. The background amounted to 1–7%, depending on the experimental conditions.

4. INSTRUMENTAL ASYMMETRY

The experimentally measured quantity was the intensity ratio of electrons scattered in the directions $\varphi = 0$ and π . Since this ratio includes the ratio of registration efficiencies, the effect of the latter was excluded by performing measurements

with an aluminum scatterer replacing the first scatterer.

Time is gained by using aluminum films made as thick as permissible, but this leads to differences between the electron spectra from gold and aluminum. With the counter windows covered by filters the registration efficiencies were very strongly energy dependent, and their ratio, because of unavoidable constructional differences in the counters also depended somewhat on energy. We must remember that the asymmetry under the experimental conditions was such that an error of 1% in the intensity ratio produced an error of 3–10% in S . Therefore the accurate elimination of the instrumental factor was an extremely laborious task. Before starting the main measurements we measured the energy dependence of the counter-efficiency ratio; when necessary, correcting filters were used to achieve the result that a 10-keV change of electron energy was accompanied by a change of at most 0.5–1% in the ratio. This determined the maximum permissible smoothing of the spectrum and thickness of the scatterer. The spectrum of electrons scattered by aluminum was measured with a magnetic spectrometer. Under the operating conditions for eliminating the instrumental factor the scatterer thickness was 1.4 mg/cm² at 245 keV, 0.55 mg/cm² at 170 and 133 keV, and 0.27 mg/cm² at 63 keV. The counting rate with aluminum was 1.5–3 times lower than with gold.

5. DEPOLARIZATION IN THE SCATTERER

The great majority of the investigations to determine S did not take into account the depolarization of the beam in the scatterers, thus incurring more or less considerable errors. In the present work, in order to exclude plural and multiple scatterings we measured S for several (usually three or four) scatterers and extrapolated the results to zero thickness. In order to reduce extrapolation errors the thinnest scatterer was such that the corresponding value of S was within 10% of the extrapolated value. In most instances this difference did not exceed 4–6%. The measurements showed that for thicknesses differing by 20–30% linear extrapolation could be used with statistical errors of 2–3%. Extrapolation required exact values of the relative scatterer thicknesses, which were obtained by comparing the scattering intensities. The absolute values were known within about 10%.

The data accumulated in our present and earlier work^[7,8] enabled us to obtain the coefficient

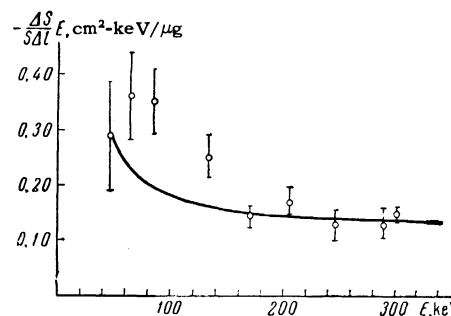


FIG. 2. Relative depolarization in gold scatterer as a function of electron energy.

α giving the dependence of relative depolarization on scatterer thickness, $\alpha = (1/S) dS/dl$, from 45 to 300 keV. The value of $|\alpha|$ decreases rapidly with increasing energy; in Fig. 2 we have for convenience plotted the product $-\alpha E$, where E is measured in keV and the thickness l included in α is measured in $\mu\text{g}/\text{cm}^2$. The data pertain to normal beam incidence on the scatterer at $\theta = 120^\circ$. The figure indicates only the statistical errors comprising 10–20%, but not including the 10% error common to all points which is associated with uncertainty regarding the absolute thicknesses of the scatterers.

Wegener^[9] has calculated the depolarizing effect of plural and multiple scattering up to quadratic terms in the thickness l . Using tables and formulas given in^[9], we calculated α numerically. It must be mentioned that, according to Wegener's work, this coefficient depends somewhat on the thickness. In the range of thicknesses for which $\Delta S/S$ varies from 0.05 to 0.3 the calculated value of α varies by not more than 10%. In Fig. 2 the continuous curve represents the calculated values of $-\alpha E$ for $\Delta S/S = 0.1$. Figure 2 shows that the calculations agree well with experiment at energies above 150 keV, while at lower energies the calculated values are ~ 1.5 times smaller than the experimental results.

6. MEASUREMENTS AND CORRECTIONS

The measurements were performed in short runs, wherein the first scatterer made of gold alternated with an aluminum scatterer. At each electron energy we used only one gold first scatterer and three or four second scatterers. As a result of the asymmetry measurements we introduced corrections of from 1 to 6% for scattering from the collodion film on which the second scatterers were deposited. The results were extrapolated to zero thickness of the second scatterer.

The angular dependence $S(\theta)$ was measured in two ways: for angles 150 – 120° with the scatterer

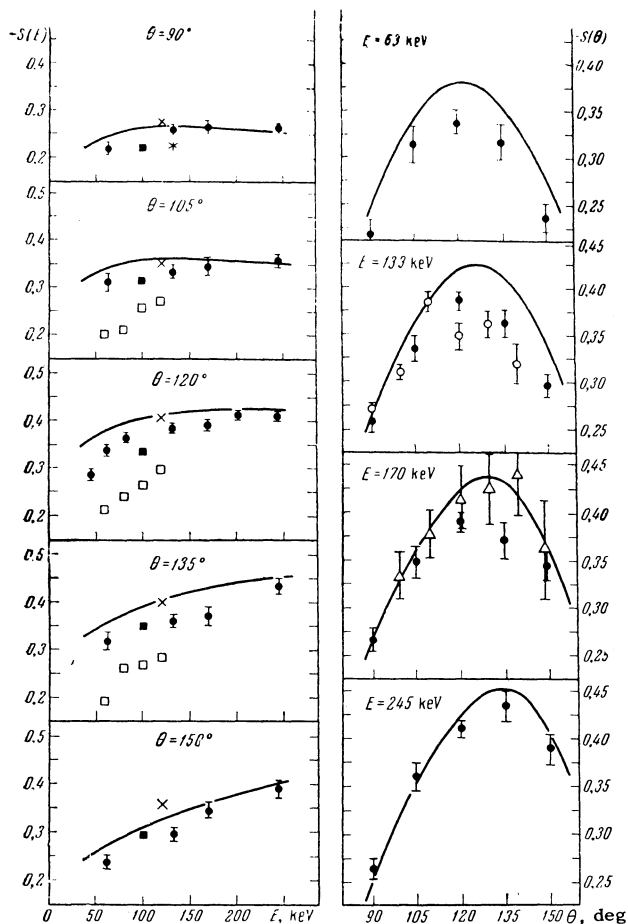


FIG. 3. Energy dependence of S . ●—present work, □—[12], *—[13]. The continuous curve was calculated^[2] for an unscreened field ($Z = 79$). ■ and × denote calculations taking account of screening in [4] and [5], respectively.

FIG. 4. Angular dependence of S . ●—present work, ○—[10], △—[14]. The continuous curve represents calculations in [2] for an unscreened field.

normal to the beam, and for $120-90^\circ$ with the beam striking the scatter at 60° . For 120° at each energy we obtained two straight extrapolation lines with somewhat different slopes; however the extrapolated values lay within the statistical errors. We used the averages of the extrapolations as our final results.

7. RESULTS AND DISCUSSION

The values of the asymmetry obtained by the foregoing procedure and extrapolated to zero thickness of the second scatterer equal $S(\theta_2)[P_{Au} - P_{Al}]$. Here P_{Au} and P_{Al} are the respective polarizations of the beam impinging on the second scatterer, independently of the angle θ_2 . To obtain $S(\theta_2)$ we used previous results given in [7] for $S(120^\circ)$.

Our results are given in the table, which contains for each angle and energy the experimental

values S_e , the theoretical values S_t obtained by interpolation of Sherman's data^[2] and converted from $Z = 80$ to $Z = 79$, and the ratios S_e/S_t . For 120° the results and errors in [7] are given; errors for the other angles combine the errors at $\theta = 120^\circ$ with those of the present work. In Fig. 3 the filled circles represent measurements of S at different energies and fixed angles, and in Fig. 4 at different angles and fixed energies. The continuous lines in these figures represent the theoretical values of S .^[2]

Figures 3 and 4 and the table show that at 245 keV the experimental values do not differ from the theory by more than 5% and in most cases only 2–4%. With decreasing energy the differences are augmented for most angles, so that at 63 keV the experimental values are 8–13% below the theoretical values; this difference becomes 20% for 120° at 45 keV. The overall effect of screening is relatively small above 60 keV; this agrees qualitatively with the calculations of Bartlett and Welton^[4] and of Mohr and Tassie.^[5] A comparison with these calculations (Fig. 3) shows that the results of Bartlett and Welton agree satisfactorily with experiment at 150° and 135° , but are somewhat too low in the interval $120-90^\circ$. Mohr and Tassie give results 5–15% higher than the experimental values, and the discrepancy increases with the scattering angle.

The differences between the experimental and theoretical dependences are small but somewhat exceed the statistical errors. At 245 keV for 90° and 105° the experimental values are somewhat higher, and for the other angles are somewhat lower, than the theoretical results. In the range 245–133 keV, S_e at $\theta = 90^\circ$ is everywhere very close to the theoretical values. At 63 keV the experimental curve duplicates the shape of the theoretical curve within experimental error and lies 11% below the latter. The angular dependence $S(\theta)$ confirms the theory as a whole with regard to the fact that with decreasing energy the maximum is shifted from 135° to 120° .

The comparison between our data and the results of other authors is sometimes hampered by the fact that the latter usually have given values of the product $S(\theta_1)S(\theta_2)$ instead of S . Since all publications on double scattering also give $S^2(\theta)$ for one of the angles, we were able to convert the results and represent $S(\theta)$ in Figs. 3 and 4; here the relative errors agree with those ascribed by other authors to the product $S(\theta_1)S(\theta_2)$. In two cases^[10,11] the electron energies differed by 10% from those in our measurements. Since S is slightly energy dependent, we considered that the

θ , deg		E, keV						
		245	204	170	133	83	63	45
90	S_t	0,255		0,265	0,267		0,247	
	S_e	$0,265 \pm 4\%$		$0,267 \pm 5\%$	$0,258 \pm 4\%$		$0,217 \pm 6\%$	
	S_e/S_t	1,04		1,01	0,97		0,88	
105	S_t	0,353		0,361	0,361		0,341	
	S_e	$0,361 \pm 4\%$		$0,348 \pm 5\%$	$0,334 \pm 4\%$		$0,313 \pm 6\%$	
	S_e/S_t	1,02		0,96	0,93		0,92	
120	S_t	0,426	0,427	0,424	0,418	0,397	0,382	0,364
	S_e	$0,411 \pm 2\%$	$0,413 \pm 2\%$	$0,390 \pm 2,5\%$	$0,385 \pm 2,5\%$	$0,365 \pm 2,5\%$	$0,337 \pm 4\%$	$0,286 \pm 4\%$
	S_e/S_t	0,96	0,97	0,92	0,92	0,92	0,88	0,79
135	S_t	0,455		0,432	0,413		0,356	
	S_e	$0,434 \pm 4\%$		$0,371 \pm 5\%$	$0,361 \pm 4\%$		$0,317 \pm 6\%$	
	S_e/S_t	0,95		0,86	0,87		0,89	
150	S_t	0,401		0,359	0,335		0,273	
	S_e	$0,389 \pm 4\%$		$0,345 \pm 5\%$	$0,295 \pm 4\%$		$0,237 \pm 6\%$	
	S_e/S_t	0,97		0,96	0,88		0,87	

results in [10,11] could be converted to our energies. The conversion factors obtained in accordance with Sherman's results did not differ from unity by more than 0.03.

The first attempt to investigate asymmetry systematically for angles greater than 90° was undertaken by Ryu et al. [12]. The latest results of this work are shown in Fig. 3. These data support the theory qualitatively, although the quantitative discrepancies are seen to be extremely large.

In 1958 Pettus [13] published asymmetry measurements for $\theta_1 = 90^\circ$ and $\theta_2 = 60^\circ, 90^\circ$, and 120° at 130 keV. He also measured the energy dependence of $S(90^\circ)S(120^\circ)$ in the range 80–190 keV. This author estimated the statistical errors at 15–40%, with the experimental results lying considerably below the theoretical values. From Pettus' results with the smallest errors we obtain a value of $S(90^\circ)$ at 130 keV lying 15% below our result.

One of the most accurate studies of double scattering is that of Nelson and Pidd, [10] who were the first to use sufficiently large apparatus, thus considerably reducing the effect of electron scattering from the walls. In addition, they placed before the counters an electrostatic analyzer for discriminating against lower-energy electrons. The angular dependence of the asymmetry $S(90^\circ)S(\theta_2)$ was measured in the range $\theta_2 = 80-140^\circ$ at 121 keV; [10] the results converted to 133 keV are shown in Fig. 4. Good agreement with our results is observed for angles smaller than 120° . For larger angles the results of Nelson and Pidd lie somewhat below ours; these authors note the possible effect of depolarization in the scatterer at such angles.

We are not aware of any other measurements of S in experiments on double electron scattering within the given energy range. Some information regarding S can be obtained from asymmetry measurements of scattered polarized electrons

emitted in β decay. Bienlein et al. [11] measured $\langle \sigma \rangle S(120^\circ)$ in the scattering of Co^{60} β electrons on gold at 209, 155, and 120 keV. They estimated their measurements to possess the very high accuracy 2–3%.

Taking for $\langle \sigma \rangle$ the value $-v/c$ given by the two-component neutrino theory, the same authors determined $S(120^\circ)$. At 209 keV their results agree with ours within experimental error. However, at 120 keV S_e/S_t equals $0.84 \pm 3\%$, which is 8% below our value. This appears to indicate that the degree of polarization of Co^{60} β electrons departs from $-v/c$. It should be noted that such discrepancies are encountered frequently; [8] therefore this procedure for determining S cannot be regarded as entirely correct.

Bienlein et al. [14] also used Co^{60} β electrons to measure the angular dependence of S at 155 keV. The results for $100-150^\circ$ converted to 170 keV are shown in Fig. 4. For large angles their results lie somewhat above ours and agree well with Sherman's calculations. At $\theta = 120^\circ$ the two experimental curves agree within experimental error.

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GAMMA RADIATION FROM HIGH-SPIN NUCLEI

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The energy spectra of the γ rays produced in the interaction of accelerated O^{16} and Ne^{22} ions with Cu, Ta, W and U nuclei were investigated in the 74–145 MeV energy range with the help of a single-crystal scintillation spectrometer. In each decay of the compound nucleus, up to 13 γ quanta with energies from 0.7 to 1.1 MeV were observed. The effect of the spin of the excited nucleus on the decay mechanism is discussed.

INTRODUCTION

THE compound nucleus produced in reactions with heavy ions is marked by a large angular momentum. At energies above the Coulomb barrier, under the assumption that any contact of the surfaces of two nuclei leads to their complete fusion with a production of a compound nucleus, the maximum angular momentum of the latter can be determined in a classical way from the relation

$$j_{\max} = \frac{R_1 + R_2}{\hbar} [2M(E - B_C)]^{1/2},$$

where R_1 and R_2 are the radii of the ion and target nuclei, M is the reduced mass, E is the kinetic energy, and B_C is the Coulomb barrier in the center-of-mass system. Thus, for example, for Ne^{20} ions with an energy of about 140 MeV in the l.s. bombarding a copper target, the compound nucleus of yttrium will have an excitation energy of 110 MeV and a maximum spin $j_{\max} \approx 70\hbar$.

The presence of such a high angular momentum in the compound nucleus is of importance for the mechanism of its decay. The theoretical investigations of the decay of rotating nuclei are very limited. The general picture of the evaporation of particles from an excited nucleus with a high angular momentum in the quasi-classical statistical approximation has been considered by Strutinskiĭ.^[1]

Strutinskiĭ obtained expressions for the mean kinetic energy and angular momentum of the evaporated neutrons. A quantitative analysis shows that in the compound-nucleus decay process practically the entire excitation energy and a comparatively small part of the initial angular momentum are carried off by nucleons. It is assumed that the residual energy and the angular momentum of the nucleus, after evaporation of the nucleons, are carried away by the cascade emission of soft γ rays.

Using the independent-particle model, Pik-Pichak^[2] also considered the evaporation of nucleons. In the case of a compound nucleus with a high rotational energy, a "centrifugal" mechanism for the emission of nucleons was predicted. Estimates made within the framework of this model also indicate that, after the emission of the nucleons, the nucleus retains a high angular momentum.

The γ radiation arising in the decay of a compound nucleus of high spin has been investigated experimentally in^[3,4].

In the present work, we studied the γ spectra resulting from the bombardment of Cu, Ta, W, and U targets by O^{16} and Ne^{22} ions in the 74–145 MeV interval.

For the $Cu + Ne^{22}$ and $Ta + O^{16}$ reactions we determined the upper limit of the cascade γ -transition lifetimes.

EXPERIMENTAL CONDITIONS

The experiment was carried out with the internal beam of the 300-centimeter cyclotron of the Nuclear Reactions Laboratory of the Joint Institute of Nuclear Research. Ions of $O^{16(+3)}$ and $Ne^{22(+4)}$, accelerated to 4–9 MeV/nucleon, impinged on a target fixed to a special probe. The beam intensity in the experiment was between 1 and 7 μA . The ion energy was varied by placing the target on orbits of different radii; for a fixed position we determined the ion energy from the stopping of ions in aluminum foils^[5] to an accuracy of at least 3%.

As targets we employed foils of Cu, Ta, W, and U of natural isotopic composition mounted on the cooled copper surface of the probe. In the experiment we used "thick" targets to avoid contamination from the background radiation arising in the

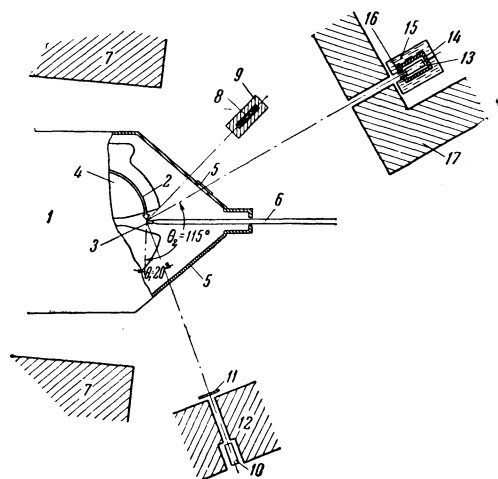


FIG. 1. Experimental arrangement. 1) Cyclotron chamber, 2) ion beam, 3) target, 4) dee, 5) diaphragm, 6) probe, 7) magnet, 8) BF_3 counter, 9) paraffin, 10) neutron counter, 11) Pb shield for γ quanta (0.02 m), 12) concrete, 13) scintillation γ spectrometer, 14) lead (0.05 m), 15) $\text{H}_2\text{O} + \text{B}$ (0.2 m) shield, 16) Pb diaphragm (0.05 m), 17) concrete shield (1.0 m)

copper base; the target thickness in the experiment was between 25 and $100\ \mu$.

The experimental arrangement is shown in Fig. 1. The flux of particles impinging on the target was measured by a sensitive detector [6] and was monitored with the aid of an all-wave BF_3 neutron counter placed close to the cyclotron chamber.

The γ radiation in the 0.05–3 MeV interval was recorded by a single-crystal scintillation spectrometer located outside the accelerator chamber at a distance of 5 m from the target. In the measurements we used a $\text{NaI}(\text{Tl})$ crystal 40 mm in diameter and 40 mm thick. Pulses from an FÉU phototube were applied to the input of a 100-channel AI-100/1 pulse-height analyzer. The spectrometer was calibrated with monochromatic γ -rays from Hg^{203} (0.280 MeV), Cs^{137} (0.660 MeV), Zn^{65} (1.11 MeV), Co^{60} (1.13 and 1.33 MeV), and Na^{24} (2.75 MeV).

To eliminate background from Compton scattering along the path from the target to the crystal, the γ -ray beam was diaphragmed twice. The absolute efficiency of the spectrometer was determined with the aid of strong calibrated γ sources of various energies placed in the position of the target. The value obtained was 4×10^{-7} . Special attention was paid to sources of background γ radiation.

During the work it was found that a large part of the background radiation over the entire range of measured γ -ray energies was due to the interaction of neutrons with the NaI crystal.

According to the experimental data, the excited compound nucleus produced in the reactions studied by us evaporated an average of 3–5 neutrons, whose spectrum has the shape of a Maxwell distribution with an energy of ~ 2.5 –3.0 MeV at the maximum.

It should be noted that to avoid distortion of the γ spectrum due to the Compton scattering on the individual components of the accelerator chamber, the γ radiation was recorded in a narrow solid angle. Shielding from scattered γ rays is "transparent" for neutrons, and the ratio of the γ -ray and neutron intensities can differ considerably from the expected value. The cross section for the inelastic scattering of 2.5-MeV neutrons is 1.96 b for iodine and 0.47 b for sodium. The γ radiation in inelastic scattering contributes mainly to the soft part of the γ spectrum ($E_\gamma \leq 0.6$ MeV).

The cross section for activation on iodine at $E_n \approx 1$ –2 MeV is small (~ 0.1 b), but sharply increases in the energy region $E_n \leq 1$ keV, attaining 50–100 b. On the other hand, it is known that each act of capture of a slow neutron by an excited I^{128} nucleus is accompanied by the emission of an average of three γ quanta of energy 2.0–2.5 MeV. In view of the fact that in our experiment the detector was at a large distance from the target, scattered neutrons, whose spectrum was considerably softer than those in the case of neutrons close to the target, could have impinged on the crystal.

In this connection we constructed shielding to protect the spectrometer from the scattered neutron flux. The phototube was placed in a lead shield (wall thickness 50 mm) which, in turn, was surrounded by a 200-mm layer of a three-percent solution of boric acid in water; in addition, a 5-mm layer of B_4C powder was placed around the crystal.

The entire apparatus was placed behind a concrete shield 1.0 m thick in which was mounted a collimator of 100-mm diameter aimed at the target. Control experiments showed that the contribution from the γ -ray background due to neutrons did not exceed 7% of the total γ -ray intensity recorded by the spectrometer. In the energy region $E_\gamma < 0.2$ MeV we observed a high counting rate of γ quanta due to electron bremsstrahlung inside the cyclotron vacuum chamber. [7]

The maximum energy of the bremsstrahlung spectrum was essentially dependent on the h. f. voltage on the dees, and hence we were faced with the problem of obtaining the necessary beam current on a given orbit of the accelerator with a minimum value of the accelerating potential.

Reaction	Max. ion energy, $E_{\max}$, MeV	Mean cross section $\bar{\sigma}_c$, b	Excitation energy E^* , MeV	Mean angular momentum $\bar{l}$, $\hbar$	Mean number of γ quanta per nucleus $\bar{\nu}$ (expt)	$\bar{E}_\gamma$, MeV	Total energy of γ radiation $\bar{\nu}\bar{E}_\gamma$, MeV
Cu + Ne ²²	74	0.36	56	19	6	0.80	5
	94	0.71	66	25	10	0.75	7
	115	0.92	75	29	13	0.70	9
	140	1.06	86	35	11	0.70	8
Ta + O ¹⁶	85	0.27	49	17	7	0.85	6
	100	0.47	61	25	9	0.80	7
W + O ¹⁶	85	0.27	50	16	7	1.05	8
	100	0.47	62	24	8	1.00	8
U + O ¹⁶	145	1.13	78	45	10 $\pm$ 2	—	—

The activation of the target and the elements of the accelerator chamber not shielded from the beam was no more than 2% of the total γ -ray counting rate during the bombardment. The background due to Coulomb excitation of the target nuclei and ion bremsstrahlung was negligible.

In a number of runs, we recorded simultaneously the γ rays and the yield of neutrons produced in the bombardment of the target by the ion beam. For this purpose we used a phototube (FÉU-33) with a stilbene crystal shielded from the cyclotron chamber by a two-meter concrete wall. A 20-mm lead plate was placed in front of the collimator to reduce the γ background of the accelerator. Reliable discrimination from γ rays was achieved for neutrons of energy $E_n \geq 3$ MeV.

EXPERIMENTAL RESULTS

1. For a beam intensity of $\sim 10^{12}$ particles/sec the γ counting rate was 100–300 pulses/sec, which allowed us to obtain a good statistical accuracy in the instrument pulse spectra. Practically all the measurements were made with the γ spectrometer at an angle $\theta = 115^\circ$ relative to the direction of the incident beam. The analysis of the instrumental spectra took into account the shape of the lines produced by monochromatic γ rays and the crystal efficiency.^[8]

2. For the Cu + Ne²² reaction we measured the γ spectra at incident beam energies of 74, 94, 115, and 140 MeV. The shape of the differential γ spectrum for two ion energies, 74 and 94 MeV, is shown in Fig. 2.

We express the absolute intensity of the radiation in terms of a parameter $\bar{\nu}$ numerically equal to the mean number of γ quanta for the decay of one nucleus. The quantity $\bar{\nu}$ was determined from the expression

$$\bar{\nu} = N_\gamma / n_0 \bar{\sigma}_c N_0,$$

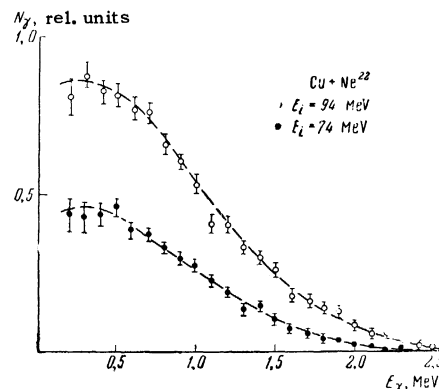


FIG. 2. Spectra of γ radiation from the Cu + Ne²² reaction measured for 74- and 94-MeV beams of Ne²² ions.

where $N_\gamma = \int_0^\infty N'_\gamma dE_\gamma / \epsilon_0 (E_\gamma)$ is the total number of γ quanta produced in the reaction; n_0 is the ion beam intensity; $\bar{\sigma}_c$ is the cross section for compound-nucleus production averaged over the target thickness¹⁾ N_0 is the effective number of target nuclei, ϵ_0 is the absolute recording efficiency of the γ spectrometer.

In each run we determined the mean energy of the γ spectrum from the expression

$$\bar{E}_\gamma = \int E_\gamma N_\gamma dE_\gamma / \int N_\gamma dE_\gamma.$$

The experimental values of $\bar{\nu}$ and $\bar{E}_\gamma$ for the investigated reactions are summarized in the table, which also includes the basic characteristics of the compound nucleus.

In view of the fact that in the estimate of the quantity $\bar{\nu}$ we used the calculated value of the cross section for compound nucleus production, the accuracy of the absolute value of $\bar{\nu}$ is difficult to estimate. The values for $\bar{\nu}$ obtained in the ex-

¹⁾In the determination of $\bar{\sigma}_c$ we used the theoretical results of Thomas^[9] on the cross section for compound nucleus production by heavy ions.

periment indicate an increase in the mean number of γ rays emitted by the nucleus as the energy of the bombarding particles is increased. These data merit an additional check, since, at energies close to the Coulomb barrier of the reaction, the energy spread of the beam ($\Delta E/E \approx 3\%$) can introduce a large error in the determination of $\bar{\nu}$. In the ion energy region $E_i > 6$ MeV/nucleon, competitive direct processes proceeding without the production of a compound nucleus will be observed.

With the given beam energy we determined, along with the measurement of the γ -ray spectrum, the relative yield of neutrons arising in the reaction. In view of the fact that compound nucleus production is a dominating process up to an energy of ~ 6 MeV/nucleon, the neutron intensity will be determined primarily by reactions of the (Ne^{22}, xn) type, where $x = x(E)$ is an increasing function of the ion energy. Then the neutron yield will increase with the energy somewhat faster than the compound nucleus production cross section.

The ratio of neutron and γ -ray intensities was measured for the $\text{Cu} + \text{Ne}^{22}$ and $\text{Ta} + \text{O}^{16}$ reactions as a function of the bombarding ion energy. The observed increase in the ratio N_γ/N_n with increasing ion energy also indicates an increase in the mean number of γ quanta emitted by the compound nucleus.

3. The study of γ radiation in the decay of a "magic" compound nucleus is of particular interest. Isotopes of "magic" nuclei of lead ($Z = 82$) could have been obtained by us in the bombardment of a W target by O^{16} ions. We compared the γ -ray spectra resulting from the bombardment of Ta and W targets by 100-MeV O^{16} ions. The ion energy was chosen close to the value of the Coulomb barrier in order to eliminate the fission process^[10] and, along with this, to decrease as far as possible the contribution of reactions involving the emission of charged particles,^[11] which could greatly distort the expected effect.

Figure 3 shows the γ -ray spectrum from $\text{Ta} + \text{O}^{16}$ and $\text{W} + \text{O}^{16}$ reactions for ion energies of ~ 100 MeV in the laboratory system. For illustration, Fig. 3 also shows the instrumental pulse distribution. It is readily seen that for "magic" isotopes of lead the mean energy of the γ spectra is approximately 1.2 times that of the neighboring Tl nuclei produced in the $\text{Ta} + \text{O}^{16}$ reaction.

4. Using the self-modulation of the ion beam in the cyclotron we performed a time analysis of the γ radiation arising in the $\text{Cu} + \text{Ne}^{22}$ and $\text{Ta} + \text{O}^{16}$ reactions. In these measurements the γ rays were recorded by a fast scintillation counter with an output to the time analyzer, which converted the

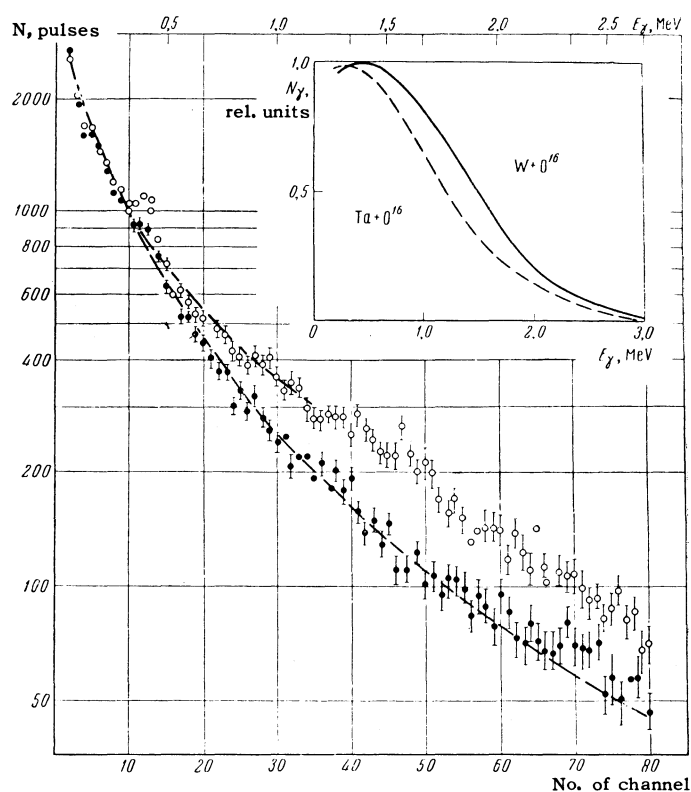


FIG. 3. Apparatus spectra from the reactions $\text{Ta} + \text{O}^{16}$ (black circles) and $\text{W} + \text{O}^{16}$ (open circles) measured for 100-MeV ions of O^{16} ; inset – corrected spectra.

time interval into an amplitude.^[12] The circuit resolution was 4.5×10^{-10} sec and the channel width was 2×10^{-10} sec.

The essence of the method was as follows. If in the bombardment of the target the radiation time is less than the duration of the current pulse, the γ -ray distribution recorded by the analyzer should closely follow the time characteristic of the internal ion beam at the given orbital radius. Figure 4 shows the time distribution of pulses obtained for the $\text{Cu} + \text{Ne}^{22}$ reaction. Measurements made on different segments of the resonance

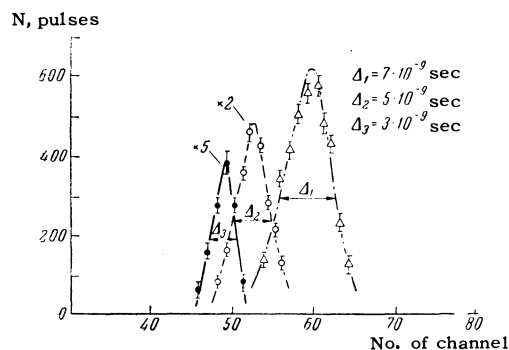


FIG. 4. Time distribution of pulses from the $\text{Cu} + \text{Ne}^{22}$ reaction for 120-MeV Ne^{22} ions measured on different segments of the beam-current resonance curve.

curve^[13] showed that the position of the peaks and the peak widths of the γ -ray distribution accurately correspond to the phase characteristics of the internal ion beam. The narrowest peak has a width of $\sim (2-3) \times 10^{-9}$ sec, and hence the upper limit of the time for the emission of cascade γ radiation is estimated to be $\tau_r < 10^{-9}$ sec.

5. To study the characteristics of the prompt γ radiation in the fission of heavy nuclei with a large angular momentum, a uranium target was bombarded by a beam of accelerated oxygen ions at an energy of 145 MeV. The measured γ spectrum was compared with the γ spectrum accompanying the fission of U^{235} induced by thermal neutrons. According to the data of^[14], each act of fission induced by the capture of a thermal neutron is accompanied by the emission of 8 ± 2 quanta, which carry away an average energy of 7 ± 1 MeV. Similar data were obtained by Leachman^[15] for the spontaneous fission of Cf^{252} .

In Fig. 5 the experimental points represent the γ spectrum from the fission of U^{238} induced by accelerated oxygen ions and the solid curve represents the prompt-radiation spectrum obtained in the fission of U^{235} induced by thermal neutrons. The calculation of the mean number of γ quanta per compound nucleus can be made in this case with greater accuracy than in the preceding measurements, since the cross section for the fission of uranium induced by heavy ions is well known,^[13] and the processes competing with fission are negligible. It follows from our data that the number of γ quanta emitted in the fission of U^{238} induced by O^{16} is $\bar{\nu} = 10 \pm 2$.

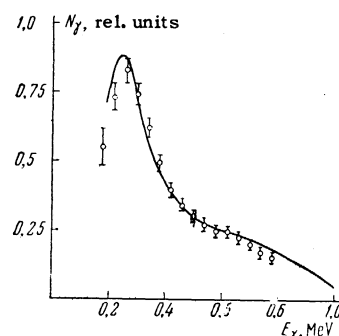
DISCUSSION OF RESULTS

From the data obtained in our experiment²⁾ it follows that when various targets are bombarded by accelerated heavy ions we observe prompt ($\tau_r < 10^{-9}$ sec) γ radiation consisting of a large number of γ quanta of mean energy 0.7–1.1 MeV.

The mean energy of γ quanta found by us for the $Cu + Ne^{22}$ and $Ta + O^{16}$ reactions is approximately 20% less than in the experiment of Mollenauer,^[4] where the γ spectra for the $Ho + C^{12}$ and $V + C^{12}$ reactions were measured at an ion energy of ~ 110 MeV. This disagreement can be removed

²⁾The characteristics of the γ radiation in the $Cu + Ne^{22}$ and $Ta + O^{16}$ reactions found in our experiment somewhat differ from the preliminary data published in^[17]. The need for a correction of the initial results arose in connection with additional experiments from which more accurate values of the level of the neutron background and the absolute recording efficiency of the γ spectrometer were obtained.

FIG. 5. Spectra of γ rays accompanying the fission of uranium induced by 145-MeV O^{16} ions (experimental points). The solid curve represents the spectrum of prompt γ rays from thermal-neutron induced fission of U^{235} .



if we take into account the contribution of double superpositions which could have occurred in Mollenauer's work, owing to the relatively high recording efficiency for the γ cascade.

Before making a suggestion regarding the mechanism of the cascade γ radiation, we compare the γ spectrum obtained by us for the excited Y nucleus (in the $Cu + Ne^{22}$ reaction) with the γ spectrum in radiative capture of a thermal neutron by the Rh^{103} nucleus^[18] (close to the Y nucleus).

Qualitative differences in the spectra are clearly seen in Fig. 6. The γ radiation arising in the $Cu + Ne^{22}$ reaction is much softer than in the radiative capture of a thermal neutron. We suggest, that, according to the hypothesis of Strutinskiĭ,^[1] this difference is due to the fact that the emission of γ quanta in the decay of a rotating nucleus occurs from states with greater angular momentum.

Under the assumption of complete statistical equilibrium, the characteristics of γ radiation from nuclei with high spin have been obtained.^[19]

Allowance for the high angular momentum of the compound nucleus gave some softening of the γ spectrum, but the obtained results cannot explain the experimental data. The cause of the disagreement is ascribed to the fact that in the process of cascade γ emission the "thermal" component of the excitation energy decreases much more rapidly than the rotational component. Owing to this, in the process of cascade γ emission the relative contribution of the rotational energy increases, and the reliability of the quasi-classical statistical approach becomes doubtful. For a de-

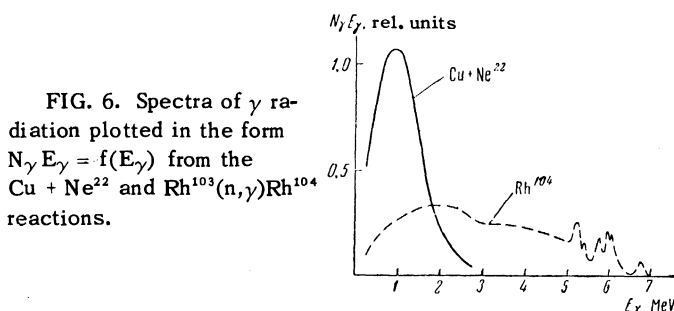


FIG. 6. Spectra of γ radiation plotted in the form $N_\gamma E_\gamma = f(E_\gamma)$ from the $Cu + Ne^{22}$ and $Rh^{103}(n, \gamma)Rh^{104}$ reactions.

tailed consideration it is necessary to have information about the structure of the levels of nuclei with high spin at excitation energies of several MeV, and this information is not available at the present time. In the special case of deformed even-even nuclei, the energy of the last quanta will apparently be determined by the γ transitions of the rotational band, which is confirmed by Morinaga's experiments (private communication) with α particles.

In the $\text{Cu} + \text{Ne}^{22}$ reaction the total energy of the γ radiation is observed to increase with the compound nucleus angular momentum. These results quantitatively explain the character of the shift in the peaks of the excitation functions toward higher energies when neutrons are evaporated from the excited nucleus in reactions with heavy ions.^[20]

When Ta and W targets are bombarded by O^{16} ions of energy 90–100 MeV, it is found that the mean energy of the γ rays in the $\text{W} + \text{O}^{16}$ reaction is approximately 1.2 times that in the case of $\text{Ta} + \text{O}^{16}$. This difference is of interest, since the excitation energy and the mean angular momentum of the compound nuclei in these reactions are practically identical.

Control measurements of the γ -ray spectra of neighboring nuclei in "crossed" reactions show that the observed effect cannot be explained by the difference in the parity of nuclei produced in the bombardment of Ta and W by oxygen ions. We believe that the difference in the character of the spectra is connected with the occurrence of a "magic" structure in the Pb compound nucleus.

Measurements of the γ -ray spectrum from the $\text{U} + \text{O}^{16}$ reaction were made to determine what happens to the initial angular momentum of the compound nucleus in the fission process. In fission the initial angular momentum appears in the relative motion of the fragments, which leads to an anisotropy in the angular distribution of the fission products.

The angular distributions of fragments from fission induced by heavy ions has been studied by a number of authors.^[21] It does not appear possible to determine from the experiments the fraction of the angular momentum which appears in the relative motion of the fission fragments. The uncertainty of such nuclear parameters as temperature, moment of inertia at the saddle point, etc. does not allow the determination of the quantitative connection between the anisotropy coefficient and the angular momentum of the excited state. At the same time, it follows from data obtained with non-

fissioning nuclei that the shape of the γ spectrum and the mean number of the emitted quanta depend on the compound nucleus angular momentum.

For the bombardment of U^{238} by O^{16} ions we obtained good agreement for the shape of the spectrum and the mean number of γ quanta with experiments on the fission of U^{235} induced by slow neutrons and in the spontaneous fission of Cf^{252} . This apparently indicates that, in the fission of nuclei induced by heavy ions, the main part of the angular momentum goes into the relative motion of the two fragments.

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PANOFSKY RATIO FOR He^3 AND THE MEAN-SQUARE RADIUS FOR THE $He^3 \rightarrow H^3$ TRANSITION

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The capture of π^- mesons stopping in He^3 has been observed for the first time. The $\pi^- + He^3 \rightarrow H^3 + \pi^0$ and $\pi^- + He^3 \rightarrow H^3 + \gamma$ reactions have been singled out, and the Panofsky ratio for He^3 (the ratio of the probabilities of the above-mentioned reactions) has been determined. Its value is used to estimate the mean-square radius for the $He^3 \rightarrow H^3$ transition.

THE capture of stopping π mesons in He^3 was investigated theoretically by Messiah^[1] and Struminskii.^[2] The conservation laws permit the following processes:

- | | | | |
|------|---|---------|---------------------|
| I. | $\pi^- + He^3 \rightarrow p + n + n$ | (55.5%) | } — absorption |
| II. | $\pi^- + He^3 \rightarrow n + d$ | (27.8%) | |
| III. | $\pi^- + He^3 \rightarrow H^3 + \pi^0$ | (9.4%) | } — charge exchange |
| IV. | $\pi^- + He^3 \rightarrow H^3 + \gamma$ | (4.8%) | |
| V. | $\pi^- + He^3 \rightarrow d + n + \gamma$ | (2.0%) | } — radiative |
| VI. | $\pi^- + He^3 \rightarrow p + n + n + \gamma$ | (0.5%) | |

The relative probabilities of these processes, given in the parentheses, were calculated by Messiah^[1] in the impulse approximation using the available data on π meson capture in hydrogen and deuterium.¹⁾ It was also assumed that the capture occurs in the S state of the mesic atom. Struminskii^[2] has shown that the ratio of reactions (III) and (IV), which we shall denote by P and call the Panofsky ratio for He^3 , can be directly related with the value of the mean-square radius of the $He^3 \rightarrow H^3$ transition by a radiative process. We have the following relation:

$$P = \frac{P_H}{1 - \frac{1}{3}k^2r^2 + \frac{1}{18}k^4r^4} \frac{\omega + M}{\omega_H + m} \frac{\omega_H}{\omega} \left[\frac{E}{E_H} \frac{M}{m} \left(\frac{\mu + m}{\mu + M} \right)^{1/2} \right], \quad (1)$$

where P_H is the Panofsky ratio, r is the mean-square radius, k is the wave number of the photon in IV, ω is its energy, ω_H is the energy of the photon in the π^- -meson capture in hydrogen, m is the neutron mass, μ is the π^0 -meson mass, M is the mass of tritium, E is the energy released in process III, and E_H is the energy released in the

process $\pi^- + p \rightarrow n + \pi^0$ (the masses of the particles and the energies are given in MeV). An experimental determination of the mean-square radius for the transition is of great interest, especially for the interpretation of data on muon capture in He^3 .

An experimental study of the π^- meson capture in He^3 has been attempted for the first time in the present experiment.

The mesons were stopped in a diffusion chamber filled with He^3 at a pressure of 20 atm.^[4] The chamber operated in a magnetic field of 6000 Oe. On the average, one stopping π^- meson was observed every four pictures. The admixture of stopping μ mesons amounted to 32% of all events. About 8500 photographs were taken in two exposures. All the material was scanned three times. The results of the scanning are as follows (the figures refer to the number of events of each type);

Single-prong stars:	
track ending in the chamber	805
track traversing the chamber	1567
total number of stars	2372
μ -e decay:	
decay electron visible	700
decay electron invisible	353
Multi-prong stars	21
π - μ decay or scattering	1423
μ -e or star (not identified)	61
π - μ or star (not identified)	33

π^- mesons produce single-prong stars only when captured in He^3 . For this reason we have to regard the detected multi-prong stars as π^- capture in the carbon or oxygen nuclei contained in the methyl-alcohol vapor.

In reactions II-IV the energy of the emitted particles is strictly fixed and, consequently, they

¹⁾The above values are corrected according to the latest, most accurate measurement of the Panofsky ratio for hydrogen $P_H = 1.533 \pm 0.021$.^[3]

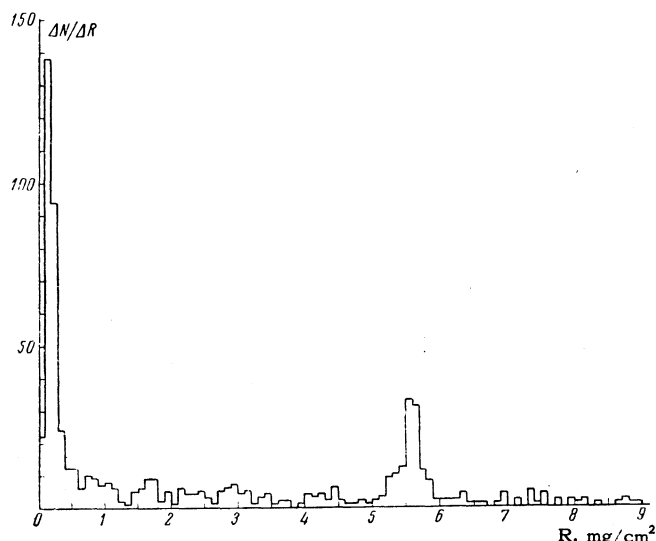


FIG. 1. Range spectrum of secondary particles in single-prong stars due to the π^- -meson capture in He^3 .

can be easily identified by measuring either their range or momentum. In the remaining processes, the energy is distributed among three or more particles. Our main concern in the present experiment was to single out the reactions III and IV. As shown above, the number of stars satisfying the selection criterion and representing, in the majority of cases, the π meson capture in He^3 , amounted to 2372. It was established that with triple scanning the detection efficiency of a π meson stopping in the chamber is independent of the nature of the event visible at the end of the meson track, and is close to 100%.

The selected events were measured with a reprojector. The range spectrum of secondary particles from single-prong stars is shown in Fig. 1. Two maxima corresponding to two monoenergetic particle groups are clearly visible at $R \sim 0.2$ mg/cm^2 and at $R \sim 5.5$ mg/cm^2 . The energy of tritium in reactions III and IV is 0.19 and 3.28 MeV, respectively. The expected values of the range of tritium having such energies correspond to the values for which the maxima have been observed in the spectrum. This indicates that the peak near 0.2 mg/cm^2 is related to the exchange process III, and the peak in the vicinity of 5.5 mg/cm^2 to the radiative capture process IV. Photographs of stars corresponding to the two processes are shown in Fig. 2. The range of the deuteron from reaction II is so large that the events could not be recorded as a star with a prong ending in the chamber. The approximately uniform background of the spectrum is partly due to reactions I, V, and VI. In selecting the charge-exchange and radiative-capture events it was assumed that the background is monotonic in the region of the peaks.

In the following we present the data concerning reaction III:

Range interval, mg/cm^2	0–0.5
Number of events	382 ± 20
Background from stopping μ mesons	23.3 ± 4.2
Background from other reactions	43 ± 7
Number of $\text{H}^3 + \pi^0$ events	316 ± 21

It can be seen from the spectrum that the secondary particles in reaction III have very small ranges. In order to achieve a high selection efficiency in the scanning we have therefore included into this group of stars, in addition to the obvious events with a short prong, also all events in which any singularity was evident at the end of the meson track, such as a kink or another non-uniformity. The background of false stars was then determined by the background of μ -meson events (scattering and others). In order to estimate the background we have used the results of the scanning of a μ meson exposure^[5] where mainly μ mesons stopped in the chamber. In that case, practically all events with a singularity at the end of the track are imitations of short-range stars because of the low π^- -meson contamination ($< 2\%$) and the small yield of stars from μ -meson capture in He^3 .

In some cases, because of a disadvantageous spatial orientation of a prong, it was impossible to measure its length (for $R < 0.3$ mg/cm^2), and some of the events (93 events) were included therefore in the range interval 0–0.5 mg/cm^2 which was not measured. The range measurements on the secondary particles from the capture reaction of thermal nucleons $n + \text{He}^3 \rightarrow \text{H}^3 + p$ and also from reaction IV showed that the deviation from the mean range is ≤ 0.3 mg/cm^2 . We can assume therefore that the majority of events representing reaction III will be contained in the range interval 0–0.5 mg/cm^2 . The detection efficiency in the chamber of prongs with such length (i.e., the probability that the prong will not be cut off by the limits of the sensitive layer) was assumed to be 100%. It was also assumed that in the range interval 0–0.5 mg/cm^2 the background from other reactions was the same as in the adjacent interval 0.5–1 mg/cm^2 .

The results concerning reaction IV are as follows:

Range interval, mg/cm^2	5.2–5.9
Number of events detected	114 ± 11
Background from other reactions	14.2 ± 1.7
Number of $\text{H}^3 + \gamma$ events divided by efficiency	146.6 ± 16.5

The detection efficiency of a prong with a length of 22 mm (5.5 mg/cm^2) was found from the analysis of the visible range spectrum of secondary particles in all stars. Its mean value (for both exposures) was found to be $(68 \pm 2)\%$. The background from

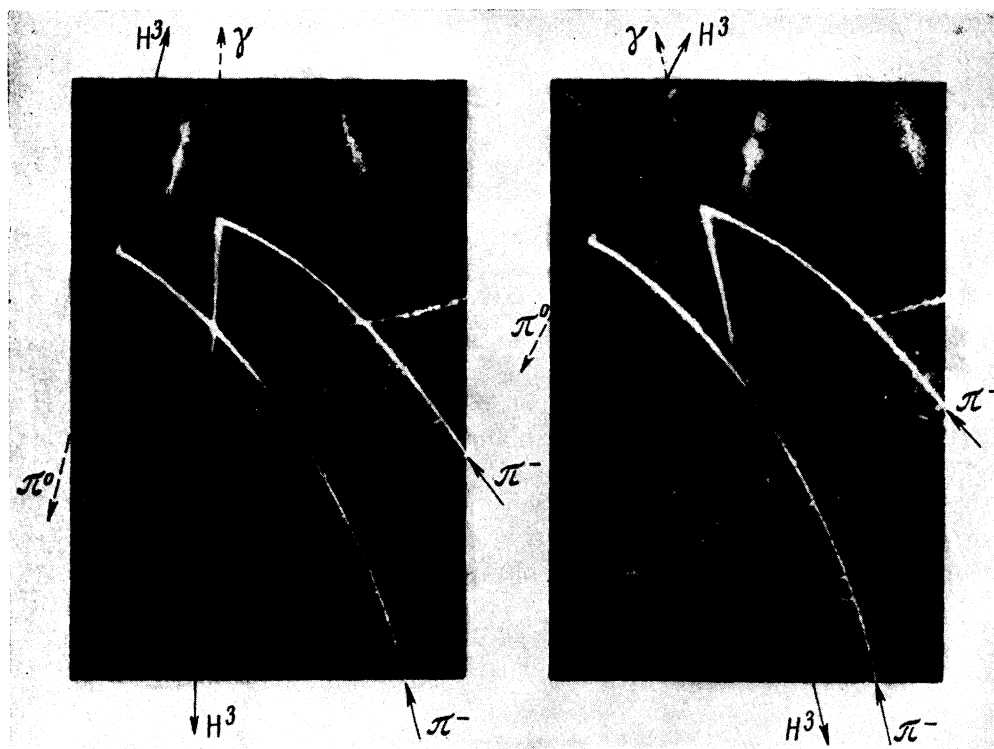


FIG. 2. Stars representing the processes $\pi^- + \text{He}^3 \rightarrow \text{H}^3 + \pi^0$ and $\pi^- + \text{He}^3 \rightarrow \text{H}^3 + \gamma$.

other reactions was determined from the intervals adjacent on the right and left of the peak associated with reaction IV. The mean range of tritium in reaction IV was found to be $5.55 \pm 0.05 \text{ mg/cm}^2$.

Using the data on reaction III and IV we find the Panofsky ratio in He^3 :

$$P = 2.16 \pm 0.28.$$

The error given is statistical only, and we have neglected the errors due to the assumptions underlying the method used to single out the charge-exchange reaction. The relative probabilities of reactions III and IV are as follows:

$$W_3 = (13.5 \pm 0.9) \%, \quad W_4 = (6.2 \pm 0.7) \%.$$

The calculation of the mean-square radius for the $\text{He}^3 \rightarrow \text{H}^3$ transition based on Eq. (1) and on the experimental value of P gives

$$r = (1.24^{+0.30}_{-0.46}) \cdot 10^{-13} \text{ cm}.$$

This value of the mean-square radius is in better agreement with the value of $1.56 \times 10^{-13} \text{ cm}$ calculated by Werntz^[6] using the known value of the binding energy of He^3 and taking the repulsive-core potential for the nucleon-nucleon interaction into account, than with the value $1.78 \times 10^{-13} \text{ cm}$ given by Fuji and Primakoff.^[7]

The yield of processes III and IV is found to be somewhat larger than that predicted by Messiah.^[1]

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ELASTIC SCATTERING OF 4.2-MeV PROTONS ON NICKEL ISOTOPES

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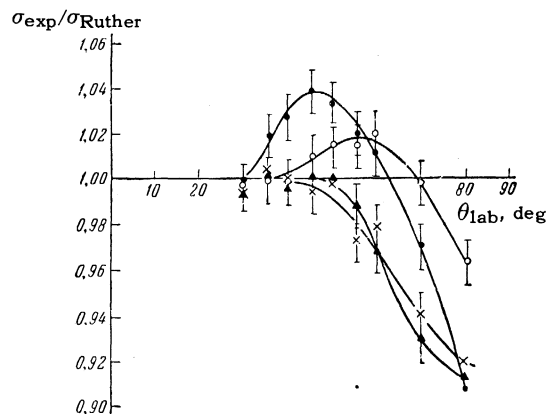
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The angular distributions of 4.2-MeV protons elastically scattered on $\text{Ni}^{58,60,62,64}$ isotopes are obtained by a comparison method. The shapes of the curves for Ni^{58} and Ni^{60} (which have sharp maxima in the angular distributions) are found to differ from those for Ni^{62} and Ni^{64} . A possible explanation of this difference is discussed.

THE angular distributions of 4.2-MeV protons elastically scattered by $\text{Ni}^{58,60,62,64}$ nuclei have been experimentally measured in a range of angles between 30° and 80° in the laboratory system of coordinates. The method used in this work has been described by us previously.^[1] The scattered protons were registered with a CsI(Tl) scintillation crystal and an FÉU-S photomultiplier. The electrical pulses of the photomultiplier were recorded with standard electronic equipment. The targets enriched to 95% were obtained by an electrolytic method and constituted self-supporting metallic foils about 1.0–1.5 micron thick. The energy resolution of the detector ($\sim 6\%$) readily permitted the segregation of inelastically scattered protons, since the energy of the first excited levels of all nickel isotopes exceeded 1.0 MeV. The angular resolution of the detector was $\pm 1.5\%$. The total error of our measurements was $\pm 1.0\%$ and was chiefly determined by the geometry of the experiment. The geometrical error occurs as a result of the wandering of the intensity of the beam of accelerated particles over the target, which leads to a change in the scattering angle.

The experimental angular distribution curves are presented in the figure in the form of the ratio of the experimental cross section to the Rutherford cross section, $\sigma_{\text{exp}}/\sigma_{\text{Ruther}}$. The curves were obtained by a comparison method consisting of a comparison of the number of particles scattered at each angle by the investigated target with the number of particles scattered by a gold target for which at the given incident proton energies the angular distribution follows the Rutherford formula.^[2] Notwithstanding the fact that the energy of the incident protons is considerably lower than the magnitude of the Coulomb barrier for nickel nuclei ($B \sim 7$ MeV), the angle dependence of the distribution of elastically scattered protons differs from the Rutherford angle dependence even at angles of 30 – 40° . We have noted previously^[3] that in the



The angular distribution of the ratio of the experimentally measured cross section to the Rutherford cross section for nickel nuclei ($E_p = 4.2$ MeV): $\circ$ — Ni^{58} , $\bullet$ — Ni^{60} , $\times$ — Ni^{62} , $\blacktriangle$ — Ni^{64} .

scattering of charged particles the competition between the Coulomb and nuclear interactions in a given energy range of the incident particles renders the differential elastic scattering cross section most sensitive to the structure of the peripheral part of the nucleus. In principle this allows us to obtain important information on the state of the surface of the investigated nuclei.

From the figure it can be seen that the character of the curves of the angular distribution of the elastically scattered protons for Ni^{58} and Ni^{60} differs considerably from the analogous curves for the Ni^{62} and Ni^{64} nuclei. It is known that a considerable difference exists between the angular distribution of protons elastically scattered by nuclei with even and odd atomic numbers,^[4] a fact explained by the considerable smearing out of the nuclear surface by the odd nucleon.

In the present work we have investigated only nuclei with even atomic number A . However, in such nuclei the excess of paired nucleons outside the closed shell can also smear out the nuclear surface very strongly.^[1,4,5] Consequently, it can

be assumed that the surfaces of the Ni^{62} and Ni^{64} are smeared out more strongly compared with the surfaces of the Ni^{58} and Ni^{60} nuclei, and the differences in the behavior of the angular distributions of protons elastically scattered by these nuclei must be explained from this point of view. Inasmuch as the angular distributions of elastically scattered protons for Ni^{62} and Ni^{64} practically coincide, it can be assumed that the difference in the threshold of the (p, n) reaction which competes with the elastic scattering of protons does not manifest itself in this range of angles at the given incident-proton energy. If the contrary were true, the curve for Ni^{64} would have to be lower than that for Ni^{62} , since the thresholds of the (p, n) reaction for Ni^{62} and Ni^{64} are 4.80 and 2.45 MeV respectively. The maxima in the angular distribution for the Ni^{58} and Ni^{60} nuclei are, apparently, due to diffraction while for the Ni^{62} and Ni^{64} nuclei no noticeably expressed diffraction aspect has been noted. It is known, however, that a diffraction pattern is distinctly observed in nucleon scattering by nuclei with a sharp boundary and having dimensions commensurate with the wavelength of the incident nucleon (the diffraction condition $\lambda < R$). This confirms the above notion on the structure of the surfaces of nickel nuclei. In addition, it can be assumed that the Ni^{60} nucleus has a sharper boundary than the Ni^{58} nucleus, since the angular distribution of the elastically scattered protons

for Ni^{60} has a sharper maximum. This is probably connected with the fact that Ni^{60} , unlike Ni^{58} , has a filled $2p_{3/2}$ neutron subshell (of the 32nd neutron).

We have measured the excitation function of elastic pp scattering for the $\text{Ni}^{58,60,62,64}$ nuclei at an angle of 120° in the energy range from 3.0–4.3 MeV. No noticeable resonances have been noted near the incident-proton energy of 4.2 MeV. It can therefore be assumed that the angular distributions of elastically scattered protons obtained in this experiment are not noticeably distorted by the effects of resonance scattering.

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INVESTIGATION OF NUCLEAR LEVELS BY NEUTRON SPECTROSCOPY. 1. NEUTRON CROSS SECTIONS OF MOLYBDENUM ISOTOPES IN THE 7–15000 eV ENERGY RANGE

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Results of measurements of the total neutron cross sections and radiative-capture cross sections of molybdenum are presented. They were obtained with the use of a linear electron accelerator and a time-of-flight neutron spectrometer. The highest resolution in the measurements of the total cross sections was $0.006 \mu\text{sec/m}$. The parameters of a number of resonance levels are determined with allowance for interference between potential and resonance scattering. The strength function $\bar{\Gamma}_n^0/D$ for each of the isotopes is also determined. Several weak levels ascribed to the p wave have been detected. The question of the identical position of levels belonging to different isotopes is discussed.

1. INTRODUCTION

DATA on the position and parameters of nuclear levels obtained thus far by neutron spectrometry techniques largely refer to a natural mixture of isotopes. However, the investigation of a number of characteristics dealing especially with the position of the levels and the distribution of the distances between them require the study of separated isotopes by neutron spectroscopy. Knowledge of the physical characteristics of separated isotopes is also necessary in a number of cases for the practical use of materials in nuclear power.

In the present paper we consider the results of the study of the total neutron cross sections and the radiative-capture cross sections for separated isotopes of molybdenum in the 7–15 000 eV interval. The choice of molybdenum for this study was based on the following considerations: a) Molybdenum has a large number of even-even isotopes, for which the relation between the position of the levels should have a particularly simple character; b) the atomic weights of molybdenum isotopes lie in a region where, according to the optical model of the nucleus, a large number of resonances connected with the capture of p neutrons should be expected; c) the data on molybdenum available in the literature^[1,2] were obtained in 1955–1956 with poor resolution; d) molybdenum is a construction material used in nuclear power.

2. APPARATUS AND SAMPLES

The measurements were made with a neutron spectrometer^[3] with a linear electron accelerator

as a pulsed source of neutrons.¹⁾

The length of the electron pulse was $0.6 \mu\text{sec}$, the channel width of the time analyzer was $0.577 \mu\text{sec}$, and the pulse repetition rate was 100 ppm.

The total cross section measurements were made with a flight distance of 109.14 m. The samples were placed 5.5 m from the target. The collimator diameter at the position of the samples (depending on the sample dimensions) was 42 or 62 mm. The neutron beam was filtered by a layer of B^{10} of thickness 0.11 g/cm^2 .

The neutron detector consisted of 230 proportional counters placed in a common aluminum tank filled to atmospheric pressure with BF_3 gas enriched to 80% B^{10} . The detector area was 2500 cm^2 and the detector thickness in the direction of the neutron beam was 17.6 cm; the maximum delay time for the occurrence of the counter pulse relative to the moment of neutron capture by boron was $0.5 \mu\text{sec}$. A special nonlinear circuit^[5] was used to apply the initial γ -ray pulse.

Information from the apparatus located in the neutron beam was fed to a 2048-channel analyzer with an electrostatic memory tube and a magnetic drum.

The beam spectrum, after subtraction of background but without correction for the detector efficiency, was approximated to an accuracy of better than 5% for the entire energy interval by the following function (E in eV):

¹⁾Preliminary results of the measurements were reported at a working conference on slow-neutron physics at Dubna.^[4]

$$N(E) = \text{const} (140/E^{1.3} + E/10^8).$$

With the diameter of the collimator aperture equal to 62 mm near the samples, the counting rate in the time-analyzer channel of width 0.577 μsec was $\sim 50 \text{ h}^{-1}$.

The radiative-capture cross-section measurements were made with a flight distance of 15.1 m. The γ -ray detector consisted of a NaI(Tl) crystal. The apparatus for the radiative-capture cross-section measurements has been described in detail in [6].

The separated molybdenum isotopes in the form of metallic or trioxide powders of molybdenum were poured into aluminum containers with an internal diameter of 50 or 70 mm and a plane-parallel bottom and cover (wall thickness 0.5–0.8 mm), packed down, and rolled in. We prepared 13 samples with an abundance of at least 75% for the principal isotope. The samples of natural molybdenum were prepared from metallic slabs 1 mm thick. The molybdenum purity in all the samples was at least 99.9%.

3. ANALYSIS OF THE RESULTS

The total neutron cross sections were determined from the attenuation of the neutron beam by the sample of the investigated material and were calculated from the usual formula:

$$\sigma_t = \frac{1}{n} \ln \frac{1}{T(E)},$$

where n is the number of atoms in 1 cm^2 of the substance and $T(E)$ is the sample transmission.

The experimental data on the total cross sections were analyzed by the area method with the aid of the well-known Hughes diagrams [7] permitting the determination of the resonance parameters with allowance for the Doppler effect and the resolution of the neutron spectrometer.

When samples of sufficiently different thickness were available, we determined the values of the radiative (Γ_γ) and neutron (Γ_n) widths. In the high-energy region, where $\Gamma_n \gg \Gamma_\gamma$, it was sufficient to use measurements with one sample; finally, in some cases for which the measured values of the areas on the transmission curves lay in the region of the Hughes diagrams where accurate values of the width could not be obtained, the analysis was made with an assumed value of the radiation width.

Additional difficulties in the analysis arose when interference between resonance and potential scattering had an appreciable effect on the shape of the resonance peak, and consequently, on the

area under the dip in the transmission curve, i.e., for neutron widths which make up an appreciable part of the total width. In these cases we used a rather simple method for preliminary analysis of the transmission curve which permitted us to take into account all factors affecting the behavior of the cross section in the resonance region, including interference, and which appreciably increased the accuracy of the determination of the resonance parameters.

The idea behind this method was to bring the resonance transmission, the deformed interference, and the overall resolving power of the apparatus into symmetric form. The sample transmission in the resonance region was, as usual, considered as a function of a dimensionless parameter

$$x = 2(E - E_0)/\Gamma$$

(E is the neutron energy, and E_0 and Γ are the position and width of the resonance level). If the energy E_0 is known, then we find the points $T(x)$ and $T(-x)$ on the transmission curve and introduce the quantity

$$T^* = \sqrt{T(x) \cdot T(-x)}.$$

The interference term is an odd function of x and vanishes here, so that by applying Hughes' method to T^* we can determine the resonance parameters.

With the use of the symmetrization procedure, the errors in the determination of the resonance parameters could, in principle, depend on the overall resolution of the apparatus and the uncertainty in the resonance energy. Numerical estimates show, however, that, in the interval of resonance parameters, sample thicknesses, and resolving power of practical importance, the error introduced by the symmetrization of the transmission curve does not exceed the errors of the area method. The method employed for data reduction has been described in detail by one of the authors. [8]

Data on the radiative-capture cross section were used in conjunction with the results of the total cross-section measurements for the identification of the levels and for the determination of the parameters of the weak levels not appearing in the total cross-section measurements. Here we used a method similar to that described by Kim et al. [9]. The total number of detector counts due to the capture of neutrons in a sample with an energy within the limits of the investigated resonance is given by the relation

$$\Sigma N_\gamma = CN(E) \sqrt{E} \varepsilon \frac{\Gamma_\gamma}{\Gamma} \int (1 - e^{-n\sigma_t}) dE = C\varepsilon N(E) \sqrt{E} \frac{\Gamma_\gamma}{\Gamma} A_T,$$

Table I

Iso- tope	E_0 , eV	Γ_γ , MeV	Γ_n , eV	Γ_n^0 , MeV	$g\Gamma_n^1$, MeV	Method of deter- mining parameters
1	2	3	4	5	6	7
Mo ⁹²	346.6±1.0		0.028±0.004	1.5±0.2		$\Gamma_\gamma = 260 \pm 80$ MeV
	2040±30					
	3170±26		12.0±1.0	210±20		$\Gamma_n \gg \Gamma_\gamma^*$
	11610±220		64.7±13.0	600±120		$\Gamma_n \gg \Gamma_\gamma$
	13920±260		61.7±12.0	520±100		$\Gamma_n \gg \Gamma_\gamma$
Mo ⁹⁴	16660±300		42.7±9.0	330±70		$\Gamma_n \gg \Gamma_\gamma$
	1519±9		1.55±0.27	40±7		$\Gamma_\gamma = 260 \pm 80$ MeV
	1688±10		0.75±0.2	18±5		$\Gamma_\gamma = 260 \pm 80$ MeV
Mo ⁹⁵	5380±60		15.3±2.0	209±30		$\Gamma_n \gg \Gamma_\gamma$
	45.0±0.4	210±60	0.175±0.010	26.1±1.5		$\Gamma_n \gg \Gamma_\gamma$
	110.8±2		(0.16±0.03)·10 ⁻³	0.0155±0.0030	78	$\Gamma_\gamma \gg \Gamma_n$
	118.3±2		(0.125±0.025)·10 ⁻³	0.0113±0.0022	52	$\Gamma_\gamma \gg \Gamma_n$
	159.0±0.3	260±50	(0.0136±0.002	1.1±0.2		$\Gamma_\gamma \gg \Gamma_n$
	220±4		(1.04±0.20)·10 ⁻³	0.070±0.014	180	$\Gamma_\gamma \gg \Gamma_n$
	249±5		(0.35±0.07)·10 ⁻³	0.022±0.004	50	$\Gamma_\gamma \gg \Gamma_n$
	267.3±0.6		(0.82±0.16)·10 ⁻³	0.050±0.010	105	$\Gamma_\gamma \gg \Gamma_n$
	335±10		(2.05±0.45)·10 ⁻³	0.112±0.025	185	$\Gamma_\gamma \gg \Gamma_n$
	358.2±1.0		0.4±0.05	21.0±2.5		$\Gamma_\gamma = 260 \pm 80$ MeV
	480±10		(5.4±1.1)·10 ⁻³	0.25±0.05	300	$\Gamma_\gamma \gg \Gamma_n$
	554.7±1.5	260±50	0.12±0.06	5±3		$\Gamma_\gamma \gg \Gamma_n$
	635±2					[12]
	679.3±2.5		0.80±0.12	30±4.6		$\Gamma_\gamma = 260 \pm 80$ MeV
	768±20					
	896±4		0.48±0.1	16.3		$\Gamma_\gamma = 260 \pm 80$ MeV
	-1022±5					
	-1147±5					
	-1413±7					
	-7400±200					
Mo ⁹⁶	114.5±2		9.0±1.0	100±11		$\Gamma_n \gg \Gamma_\gamma$
	-131.1±0.2	260±50	(0.75±0.25)·10 ⁻³	0.070±0.023	340	$\Gamma_\gamma \gg \Gamma_n$
	-2371±17		0.200±0.015	17.3±1.3		$\Gamma_n \gg \Gamma_\gamma$
	-3300±30		5±0.5	10.3±1		$\Gamma_n \gg \Gamma_\gamma$
	-3660±30		14.8±2.0	260±35		$\Gamma_n \gg \Gamma_\gamma$
Mo ⁹⁷	70.9±0.15	330±80	0.0175±0.0017	2.06±0.20		$\Gamma_\gamma \gg \Gamma_n$
	210±6		(0.78±0.16)·10 ⁻³	0.054±0.011	145	$\Gamma_\gamma \gg \Gamma_n$
	230±6		(2.05±0.45)·10 ⁻³	0.135±0.030	330	$\Gamma_\gamma \gg \Gamma_n$
	267.3±0.6					
	285.4±0.7		0.064±0.012	3.8±0.7		$\Gamma_\gamma = 260 \pm 80$ MeV
	311.7±1.0		(8.0±2.0)·10 ⁻³	0.45±0.11	800	$\Gamma_\gamma \gg \Gamma_n$
	396.7±1.2	260±80	0.080±0.040	4±2		$\Gamma_\gamma = 260 \pm 80$ MeV
	505±3		0.53±0.1	23±4		[12]
	576±2	260±80	0.67±0.13	28±6		[12]
	695±20					
	788±3		2.15±0.20	76±7		$\Gamma_\gamma = 260 \pm 80$ MeV
	1255±6		3±0.3	8.1±0.8		$\Gamma_\gamma = 260 \pm 80$ MeV
	1519±9					
	2000±20					
Mo ⁹⁸	12±0.1	200±6	(0.07±0.005)·10 ⁻³	(0.02±0.0014)		
	429.7±1.3		0.070±0.010	3.4±0.5	900	analysis from shape
						$\Gamma_\gamma = 260 \pm 80$ MeV
	467.0±1.4	260±50	0.70±0.08	32±4		two-sample technique
	612±2		0.035±0.005	1.4±0.2		$\Gamma_\gamma = 260 \pm 80$ MeV
	816±3		0.080±0.015	2.8±0.5		$\Gamma_\gamma = 260 \pm 80$ MeV
	1103±5					
	1519±9		1.40±0.21	36.0±5.4		$\Gamma_\gamma = 260 \pm 80$ MeV
	2550±20		1.44±0.20	29±3		$\Gamma_\gamma = 260 \pm 80$ MeV
	3300±30		4.0±0.3	70±5		$\Gamma_n \gg \Gamma_\gamma$
Mo ¹⁰⁰	9000±130		50±5	30±50		$\Gamma_n \gg \Gamma_\gamma$
	99.5±0.2		(0.56±0.22)·10 ⁻³	0.056±0.022	320	$\Gamma_\gamma \gg \Gamma_n$
	363±1	260±80	1.00±0.12	52±6		[12]
	1067±5		0.42±0.08	13±2.5		$\Gamma_\gamma = 260 \pm 80$ MeV
	1255±6		0.315±0.030	9.0±0.9		$\Gamma_\gamma = 260 \pm 80$ MeV
	1668±10		1.15±0.10	28±3		$\Gamma_\gamma = 260 \pm 80$ MeV
	1936±12		3.34±0.20	76±5		$\Gamma_n \gg \Gamma_\gamma$

* $\Gamma_\gamma \gg \Gamma_n$ are the weak levels which can be ascribed to the p wave. In these cases we have introduced the parameters $g\Gamma_n$ and $g\Gamma_n^0$ obtained from measurements of the capture cross section under the assumption that $\Gamma_\gamma = 260 \pm 80$ MeV.

**In these cases the values of the parameters were taken from [11] and the position of the resonances is based on our measurements.

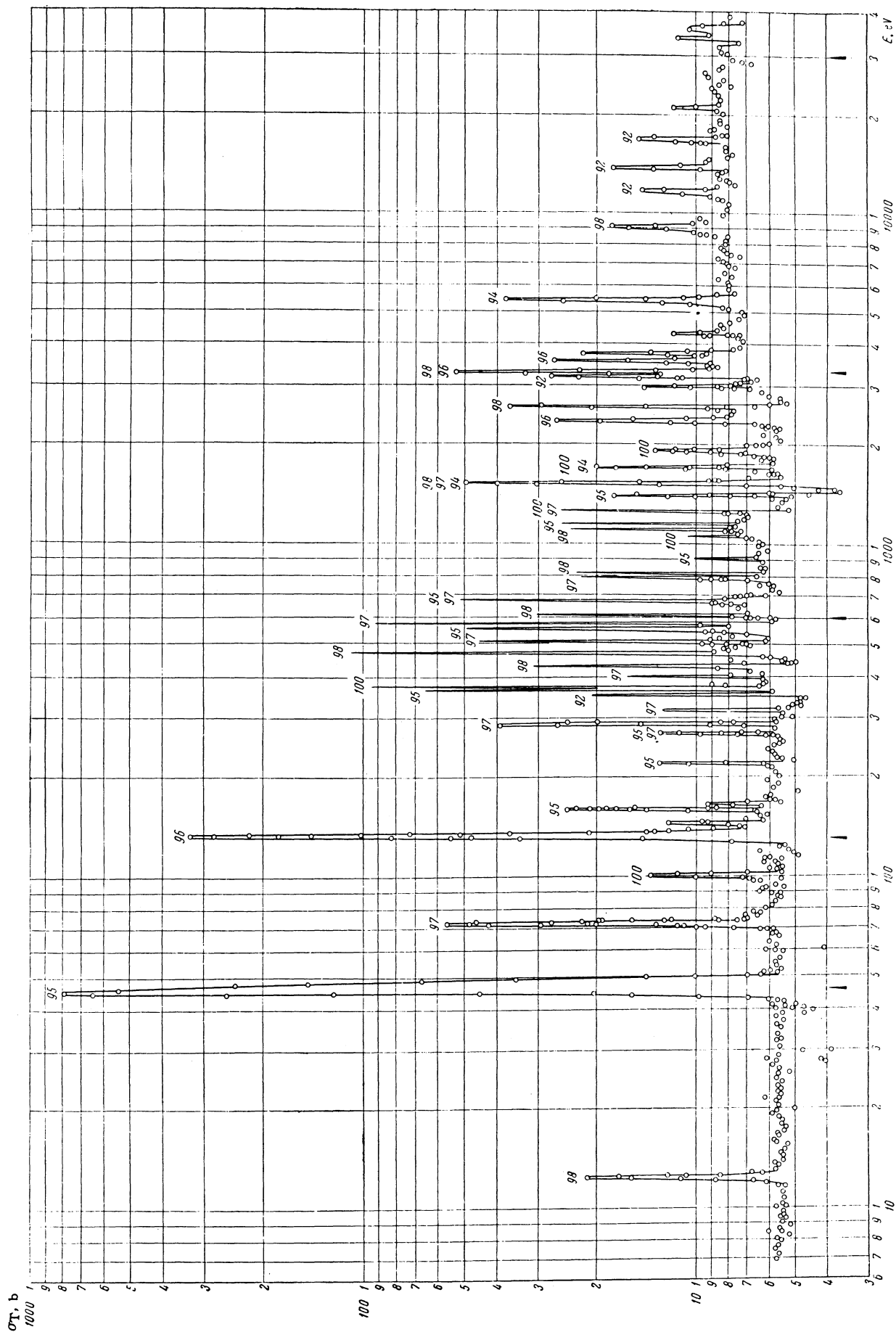
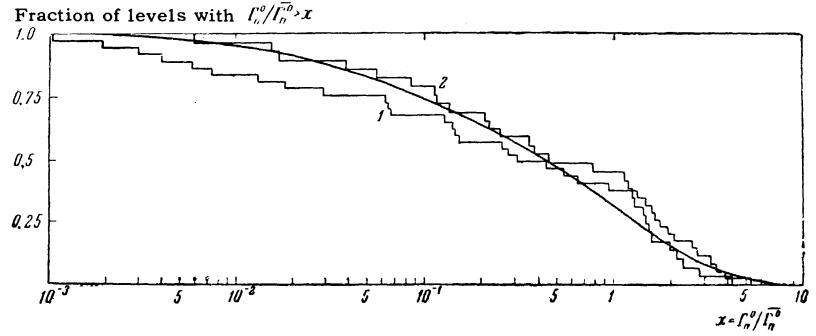


FIG. 1. Total neutron cross section for natural molybdenum.

FIG. 2. Neutron width distribution.



where ϵ is the recording efficiency of the detector for neutron capture, N_γ are the counts in the time-analyzer channel, $CN(E)\sqrt{E}$ is the total intensity of neutrons falling on the sample during the measurements ($C = \text{const}$), and A_T is the area in the dip of the transmission curve for the investigated sample.

Applying this relation to the resonance whose parameters are known from the total cross-section measurements, we can determine the quantity $C\epsilon$. For this purpose, we used the parameters of the levels of Mo^{95} at 159 eV, of Mo^{97} at 70.9 and 285 eV, and of Mo^{96} at 131 eV. Knowing the magnitude of $C\epsilon$ and taking Γ_γ to be the same for all levels, we can determine the value of A_T for the weak levels ($\Gamma_\gamma/\Gamma \approx 1$) from the radiative-capture measurements, after which the value of $g\Gamma_n$ could readily be determined from the calculated curves of A_T/Δ as functions of $n\sigma_0$ and Γ/Δ (see reference 10). The quantity $C\epsilon$ and the parameters $g\Gamma_n$ determined in this way for the weak levels depend very weakly on the value taken for Γ_γ , namely, $\Gamma_\gamma = (260 \pm 80)$ MeV.

4. RESULTS AND DISCUSSIONS

The results of the measurements and the values of the level parameters obtained from them are given in Table I.

Besides the molybdenum levels shown in Table I, we also observed levels lying at 967 ± 4 , 1200 ± 6 , 1582 ± 9 , 1761 ± 10 , and 3760 ± 34 eV, which we were unable to assign to definite isotopes.

Figure 1 shows the behavior of the total cross section for natural molybdenum.²⁾ The resonances detected only molybdenum in good agreement with our data. We take this opportunity to thank the authors for sending the preprint.

In Fig. 2 (step-like curve 1) we show the dis-

tribution of the neutron widths for 37 levels of Mo^{95} , Mo^{97} , Mo^{98} , and Mo^{100} . The Mo^{95} level at 7400 eV and the Mo^{98} level at 9000 eV have been omitted. The smooth curve corresponds to the Thomas-Porter distribution for one degree of freedom. We note that there is a large deviation of the experimental points from this curve, especially in the region of small neutron widths.

The agreement is considerably improved if we eliminate the weakest levels from the experimental data (Mo^{95} at 110.8, 118.3, 220, 249, 267.3 eV; Mo^{97} at 230 eV, Mo^{98} at 12 eV, and Mo^{100} at 99.5 eV) by considering them to correspond to the capture of p neutrons. The data for the remaining 29 levels are also shown in Fig. 2 (step-like curve 2).

Table II lists the mean distances D between the s levels for each isotope.

For Mo^{95} and Mo^{97} we list the mean distances calculated for a one-spin state under the assumption that the density of the two systems of levels is the same.

Since the binding energies of the nucleon for molybdenum isotopes differ by approximately 2 MeV, we also calculated for the different isotopes the quantities D^* reduced to one excitation energy of 6.5 MeV (see Table II). The quantity D^* proved to be largest for Mo^{92} , which has a closed neutron shell and decreases with increasing atomic weight.

In the last row of Table II we list the values of the strength function for s-waves $S_0 = \overline{\Gamma_n^0}/D$ for the isotopes studied. It should be noted that the obtained data, which have large errors, lie appreciably above the results of other experiments^[13] for this region of atomic weights and are in better agreement with the predictions of the optical model of the nucleus.

For the case of Mo^{95} we also calculated the strength function S_1 for p-waves and obtained the value $(6.3 \pm 4) \times 10^{-4}$. The calculations were made with the formula

$$S_1 = \frac{1}{2L+1} \frac{\langle g\Gamma_n^1 \rangle}{D_1} = \frac{1}{3\Delta E} \Sigma g\Gamma_n^1,$$

²⁾After submission of this paper, we received a preprint from Saclay^[12] which gave data on the position of the resonance levels for natural molybdenum in good agreement with our data. We take this opportunity to thank the authors for sending the preprint.

Table II

Isotope mass number	92	94	95	96	97	98	100
D , eV	2400 ± 1000	1000 ± 400	220 ± 50	1000 ± 500	220 ± 50	270 ± 80	430 ± 150
D^* , eV	7500	1460	1170	1120	570	83	67
$S_0 \cdot 10^4$	1.0 ± 0.7	0.5 ± 0.4	0.55 ± 0.4	0.8 ± 0.7	0.6 ± 0.3	0.8 ± 0.5	0.9 ± 0.6

where $g\Gamma_n^1 = g\Gamma_n E^{-3/2} E_R$, $\Delta E = 350$ eV is the energy interval in which the p levels were observed, E_R is the energy at which the neutron wavelength is equal to the radius of the nucleus, and Γ_n^1 is the reduced neutron width for p levels.

Attention is drawn to the fact that the positions of the resonances belonging to different isotopes of molybdenum coincide. Thus we note the agreement between the levels of Mo^{95} and Mo^{98} at 470 ± 3 eV, Mo^{95} and Mo^{97} at 685 ± 6 eV, Mo^{98} and Mo^{100} at 1085 ± 15 eV, Mo^{97} and Mo^{100} at 1255 ± 7 eV, Mo^{94} and Mo^{100} at 1680 ± 10 eV and also three-fold agreement for the levels of Mo^{92} , Mo^{95} , and Mo^{100} at 355 ± 10 eV and Mo^{94} , Mo^{97} , and Mo^{98} at 1520 ± 10 eV. The presently obtainable resolution and statistical accuracy do not permit the reliable resolution of all cases of coinciding levels belonging to different isotopes.

Coinciding levels belonging to different isotopes have also been observed earlier, for example, for the case of nickel isotopes by Newson et al.^[14] It is of interest to estimate the probability of a random coincidence of levels of different isotopes. The expected number α_2 of chance coincidences of two levels in an energy interval ΔE can be estimated from the formula

$$\alpha_2 = E \Delta E \sum_{i \neq k}^m \frac{1}{D_i D_k}$$

and for a three-fold coincidence we have

$$\alpha_3 = E \Delta E^2 \sum_{i \neq k \neq l}^m \frac{1}{D_i D_k D_l},$$

where E is the considered energy interval; D_i , D_k , D_l are the mean distances between levels for the i -th k -th, and l -th isotope; and m is the number of isotopes considered.

In our case $m = 7$, and, setting $E = 2 \times 10^3$ eV and $\Delta E = 20$ eV, we obtain $\alpha_2 = 4.5$ and $\alpha_3 = 1.4$. Eleven double and two triple coincidences have been observed experimentally in the given energy interval. The available experimental data are insufficient for an unambiguous answer to the question of whether there exists an effect involving the coincidence of levels of different isotopes.

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MAGNETOELECTRIC EFFECT IN TITANIUM OXIDE Ti_2O_3

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The magnetoelectric effect has been discovered in polycrystalline Ti_2O_3 . Crystallographically this compound is isomorphous with Cr_2O_3 , in which the effect has been previously observed.^[1,2] The most probable type of the magnetic structure of Ti_2O_3 was determined.

AN experimental observation of the magnetoelectric effect in chromium oxide Cr_2O_3 has been reported earlier.^[1,2] The magnetoelectric effect^[3,4] is manifest in the appearance of magnetization in a sample placed in an electric field, or in the appearance of electric polarization in a magnetic field. This effect arises from the fact that a magnetic structure of IR type (here I is an inversion transformation and R is an operation consisting of simultaneous change of the directions of magnetic fields and spins) allows the existence of terms proportional to EH in the thermodynamic potential. The magnetic symmetry of Cr_2O_3 includes an element of IR type, in contrast to other substances with known magnetic structure having the symmetry element I which forbids the magnetoelectric effect.

It was of interest to investigate the magnetic properties, and in particular the magnetoelectric effect, of substances with unknown magnetic structure, but having crystal structure similar to Cr_2O_3 . V_2O_3 and Ti_2O_3 were selected for the tests. Their crystal structures are the same as Cr_2O_3 ; the unit cell parameters are close to the parameters of Cr_2O_3 .^[5] The magnetic properties of V and Ti atoms are similar to the properties of Cr. From the work of Hoshek and Klemm^[6] and Foëx^[7] it is known that a phase transition, presumably antiferromagnetic, occurs in V_2O_3 at 170°K. The existence of the antiferromagnetic state in Ti_2O_3 with a transition at 250°K was first noted in the work of Adler and Selwood^[8] and Naylor,^[9] but it was not detected by Pearson.^[10]

MEASUREMENTS ON V_2O_3

For these tests we used single-crystal samples grown by the Verneuil method and kindly supplied by A. A. Popova, a member of the staff of the Crystallography Institute. The samples were made

spherical by grinding with carborundum. The sample diameter was 2–5.5 mm. Debye diffraction patterns were used to check stoichiometry. The deviation from the stoichiometric composition did not exceed 1%.¹⁾

The susceptibility of V_2O_3 was investigated using magnetic balances described earlier.^[11] It was found that the temperature dependence of the magnetic susceptibility differed from the curve described by Hoshek and Klemm^[6] by a much stronger discontinuity and by the susceptibility below the transition point being independent of temperature within the experimental error right down to 14°K. A marked hysteresis was found during the measurements, which definitely indicated a first-order transition. At this transition there was apparently a discontinuous change of the volume since, on rapid variation of temperature in this region, the single crystal cracked or even transformed into a coarse powder.

An attempt to detect the magnetoelectric effect by the technique used for Cr_2O_3 gave a negative result. After these measurements the authors became acquainted with the work of Paoletti and Pickart^[12] on the determination of the magnetic structure of V_2O_3 by neutron diffraction. It became clear that at the phase transition V_2O_3 changes its crystal symmetry and the magnetoelectric effect should no longer exist in it.

MEASUREMENTS ON Ti_2O_3

Measurements were carried out on polycrystalline samples of Ti_2O_3 prepared by reduction of the oxide TiO_2 .²⁾ Foreign impurities in the samples did not exceed 0.1% and the deviation from

¹⁾The authors are grateful to M. A. Gurevich for obtaining and interpreting the Debye diffraction patterns.

²⁾The authors take this opportunity to thank L. A. Mikhel'son for preparing the oxide.

stoichiometry was also less than 0.1%. Polycrystalline samples were used because of the difficulties in preparing single crystals. Titanium easily forms compounds with oxygen with a great variety of compositions. Using the Verneuil method one can hardly obtain a single crystal of Ti_2O_3 which melts at $\approx 1900^\circ\text{K}$ and transforms into other oxides even at $1300\text{--}1400^\circ\text{C}$.

The technique for detecting the magnetoelectric effect was very simple.^[2] However, it should be remembered that the first attempt^[2] to detect this effect in polycrystalline Cr_2O_3 ended in failure, because of the presence of two possible orders of spin distribution in the unit cell, corresponding to opposite signs of the magnetoelectric effect. Nevertheless when the antiferromagnetic transition of polycrystalline Cr_2O_3 took place in electric and magnetic fields, it was possible to detect a magnetoelectric effect equal in magnitude to about one-third of the effect in a single crystal along the three-fold axis. The electric-field uniformity was ensured by placing the sample in a teflon bath filled with dichloroethane. This result for Cr_2O_3 suggested that it should be feasible to investigate a polycrystalline sample of Ti_2O_3 .

To detect the magnetoelectric effect in Ti_2O_3 we used the method of applying an alternating magnetic field and measuring the potential difference at the ends of the sample. The choice of this technique was dictated by the considerable electrical conductivity of Ti_2O_3 ,^[13] which made it difficult to establish a strong electric field in the sample.

In selecting the alternating-field frequency we took into account the fact that the electrical conductivity of the substance would cause rapid neutralization of the electric charge appearing at the ends of the sample under the action of the magnetic field H . At low frequencies, when the sample dimensions are small compared with the depth of the skin layer and the magnetic field in the sample is uniform, we can obtain the following dependence for the voltage V at the poles of a sphere:

$V \propto (a\omega/\sigma)\alpha H$ where a is the radius of the sphere, σ is the electrical conductivity, α is the magnetoelectric effect constant, and ω is the frequency. From this dependence it is evident that to obtain high voltages one should use high frequencies. However, with increase of the frequency the skin-layer depth becomes smaller than the sample dimensions, and the magnetic field begins to act only in a surface layer of thickness decreasing rapidly with increasing frequency. This reduces the voltage V . The selected frequency was 4 Mc and corresponded to a skin-layer depth of ≈ 3 mm.

The high-frequency magnetic field intensity was 1 Oe.

The sample, compacted by a pressure of ≈ 2500 atm into a cylinder of 5 mm diameter and 8–10 mm length, was provided with electrodes at its end and placed in a coil so that the sample axis coincided with the coil axis. The electrical resistance of the samples depended strongly on the compacting technique and varied within the limits of one order of magnitude at room temperature.

The temperature dependence of the voltage at the sample ends was measured in the temperature interval from 20°K to room temperature. The required temperatures were obtained by placing the whole device into a test tube surrounded by a vacuum jacket placed in turn in a Dewar flask. Provision was made to take the sample through the antiferromagnetic point in a magnetic field; for this purpose a solenoid was used which produced a constant field of up to 3000 Oe intensity. The sample temperature was measured with a copper–constantan thermocouple.

The tuned amplifier used in this work had a noise level of $1.5 \mu\text{V}$ at the input. To obtain the highest possible high-frequency field the solenoid was constructed in a special way. It was wound with one layer of ribbon 0.1 mm thick and 2.2 mm wide. The plane of the ribbon was perpendicular to the solenoid axis.

The results of the measurements are given in Fig. 1. Curve 1 represents the temperature dependence of the signal at the sample electrodes at a constant amplitude of the alternating magnetic field. Throughout the test range of temperatures the dc electrical resistance was also recorded (curve 2). The figure shows clearly a singularity on curve 2 at a temperature corresponding to the antiferromagnetic transition in the sample. It is also seen that the voltage at the sample electrodes at 250°K begins to increase smoothly, reaching a maximum at 165°K . The position of the electrical signal maximum and the transition temperature were identical within the experimental error for all four samples used in tests, irrespective of the ohmic resistance of these samples.

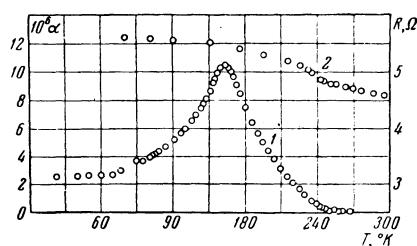


FIG. 1

It should be noted that because of a stray capacitance between the solenoid and the sample there was a certain induced noise signal at the amplifier input, which usually did not exceed 50% of the difference between the signals at the maximum and at the transition point. In some measurements the induced noise signal at room temperature was compensated by a device which allowed variation in the required way of the amplitude and phase of the voltage of the oscillator which supplied the coil. In other measurements the magnitude of the induced noise signal at room temperature was simply subtracted from the total voltage. The resultant curves coincided within the experimental error. From our measurements the value of α , the coefficient of proportionality between the electric polarization and the applied magnetic field, was found to be $\alpha_{\text{max}} = 10^{-5}$.

Six tests were carried out in which the sample was taken through the transition temperature in an external magnetic field with several volts applied across the sample electrodes. It was found that the amplitude of the signal at the electrodes (curve 1) was always the same irrespective of the parallel or antiparallel orientation of $\mathbf{E}$ and $\mathbf{H}$, and that it was identical with the value obtained at the transition when no external electric or magnetic field was applied to the sample. It should be noted that the sign of the electric signal was always the same and that the crystal behaved as a single domain.

On the basis of these tests we may conclude that the magnetoelectric effect exists in Ti_2O_3 but that it is weaker than in chromium oxide. Thus the thermodynamic potential of Ti_2O_3 contains terms with the product of $\mathbf{E}$ and $\mathbf{H}$, and the order of alternation of the spin directions of the Ti ions in a unit cell is that shown in Fig. 2. As regards the spin orientation with respect to the crystallographic axes, we can only say that it is much more probable that the spins in a single-domain crystal lie in the basal plane than along a three-fold axis (as occurs in chromium oxide).

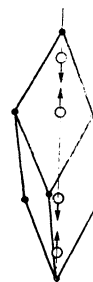


FIG. 2

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NEUTRON TRANSFER BY THE Be^9 NUCLEUS

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The mechanism of Be^8 production in interactions of Be^9 ions of maximum energy 85 MeV with silver and bromine nuclei of nuclear emulsion has been investigated. The angular distribution of Be^8 produced in the ground state has two pronounced peaks in the small-angle region; on the other hand, the angular distribution of Be^8 in the excited states has only a single peak at smaller angles. The Be^8 yield was obtained as a function of the Be^9 ion energy. The collision parameter of the interacting nuclei involved in the formation of the Be^8 nucleus in the ground state is estimated and the value $r_0 \approx 1.95 \times 10^{-13}$ cm is deduced. It is shown that the principal mechanism of Be^8 formation is neutron transfer.

IN the study of interactions of Be^9 ions with emulsion nuclei, the production of Be^8 was rather frequently observed. The Be^8 nucleus is unstable relative to decay into α particles and gives rise, in emulsion, to a characteristic track terminating in two prongs, from which this nucleus can be readily identified.

Owing to energy and momentum conservation, the energies E_1 and E_2 of the α particles produced as a result of the Be^8 decay and the angle between them $\theta_{1,2}$ in the l.s. are related as follows:

$$E_1 + E_2 - 2\sqrt{E_1 E_2} \cos \theta_{1,2} = 2Q, \quad (1)$$

where Q is the excitation energy of the Be^8 nucleus. From the angles and energies of the α particles we can readily obtain the angle and energy of the Be^8 nucleus.

Most likely, the main process leading to the production of Be^8 under these conditions is the process of neutron transfer, although the disintegration process can also make a certain contribution. Since the neutron transfer reactions proceed on the surface of the nuclei, the study of these reactions is of definite interest, since it can be hoped that it will assist us in studying the conditions on the nuclear surface.

Pellicles of NIKFI-D nuclear emulsion 300–400 μ thick were exposed to Be^9 ions accelerated to an energy of 85 MeV in the linear heavy-ion accelerator of the Physico-technical Institute of the Ukrainian Academy of Sciences. The ions entered the emulsion at an angle of 25° to the surface. In the emulsion, α particles could be reliably distinguished from protons.

We selected 284 two-prong stars corresponding

to a reaction of the type $(\text{Be}^9, \text{Be}^8)$ on silver or bromine nuclei. Of these, 146 corresponded, according to relation (1), to the production of Be^8 in the ground state ($Q = 0.1$ MeV, $\Gamma \approx 50$ eV) and 138 corresponded to the production of Be^8 in excited states. The distribution of the Be^8 excitation energies obtained from calculation by relation (1) is shown in Fig. 1.

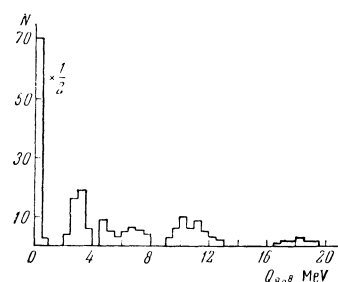


FIG. 1. Distribution of Be^8 nuclei for the energy states Q obtained from two-prong stars according to (1).

The first group on the left corresponds to the production of Be^8 in the ground state. The second and fourth groups correspond to the production of Be^8 in excited states with an excitation energy of $Q = 3$ MeV ($\Gamma = 1.46$ MeV) and $Q = 11.5$ MeV ($\Gamma \approx 6$ MeV), respectively. The fifth group corresponds to energy levels situated close to one another. However, in view of the isospin selection rules, these energy states cannot be obtained by means of the same mechanism as in the case of the other states. Evidently, we have to do here with another mechanism. This group was not included in the considerations that follow. In the 5–8 MeV excitation energy interval, where a third broad group is found, the Be^8 energy level is unknown. Since the initial ion energy was known,

then, from the measurements of the Be^9 ion range, we could determine the energy at which the reaction occurred.

The angular distributions of Be^8 produced in the ground and excited states (for the entire interval of Be^9 ion energies) are shown in Figs. 2a and 2b, respectively. Owing to the poor statistics, we were not able to construct the angular distribution separately for each excited state.

In view of the fact that it was impossible to determine whether the reaction occurred on a Ag or Br nucleus, we used a hypothetical nucleus with $Z = 41$ and $A = 94$ for transformation to the c.m.s. and in other calculations.

The angular distribution of the Be^8 nuclei produced in the ground state has two pronounced peaks in the small-angle region. The angular distributions of Be^8 for two energy intervals of Be^9 ions, less than and greater than 65 MeV (not shown in the figures), have the same structure with a slight broadening and shift of the second peak toward the larger angles as the Be^9 energy ion decreases.

The occurrence of two peaks in the angular distribution of Be^8 in the ground state indicates the existence of two mechanisms for the production of this nucleus. In accordance with the theoretical considerations of Breit and Abel,^[1] the peak at large angles is apparently due to the tunnel mechanism of neutron transfer, which has the greatest probability when the bombarding ion is as close as possible to the target nucleus without yet penetrating the Coulomb barrier.

The peak at smaller angles is apparently due to the grazing interaction^[2], where the bombarding ion penetrates the Coulomb barrier and interacts with the attractive potential of the target nucleus. The attractive potential counteracts the repulsion due to the Coulomb field of the nucleus and decreases the resulting deflection. This leads to a

peak in the angular distribution at small angles.

In the angular distribution of Be^8 produced in excited states (Fig. 2b) only one peak is observed at small angles. It is apparently due mainly to grazing interactions, although it is not excluded that it is partly the result of the tunnel mechanism. Perhaps this is an indication of a combined effect due to several excited states.

Figure 3 shows the relative energy loss in the interaction. In the case of Be^8 production in the ground state, this nucleus retains 85–90% of the initial ion energy. In the production of Be^8 in excited states, the distribution of the relative energy loss is considerably broader. This is evidence of the more intensive exchange of energies between the interacting nuclei, as a result of which both nuclei go into strongly excited states.

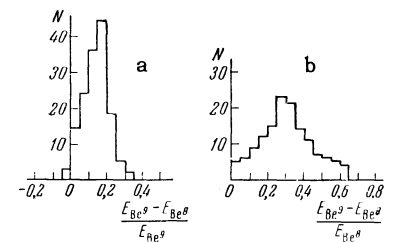


FIG. 3. Relative energy loss in interactions with Be^8 production in the ground state (a) and Be^8 production in excited states (b).

It is probable that when Be^8 is produced in the ground state, the Ag or Br nucleus remains in the ground state or in low-lying excited states, whereas the production of Be^8 in excited states proceeds with a much greater energy exchange between the interacting nuclei, as a result of which the Ag or Br nucleus also remains in a strongly excited state.

In the plot of the energy dependence of the excitation function (Fig. 4), we observe a sharp change in behavior for Be^9 ion energies of 55–60 MeV, and with a further increase in the energy the cross section for the neutron transfer reaction no longer increases. Such a behavior of the excitation function can be explained in the following way. Above the energy threshold, the probability for neutron transfer increases rapidly with increasing energy of the bombarding ion until the interaction distance reaches the absorption radius R_a . This corresponds to a rapid increase in the excitation function above threshold. With a further increase in energy, that is, as the interacting nuclei approach closer to each other, the probability for the neutron transfer process begins to drop rapidly, since for interaction distances less than the radius R_a , the process in which Be^9 is absorbed by silver or bromine starts to compete strongly. This is reflected by the change in slope and the smoothing of the excitation function.

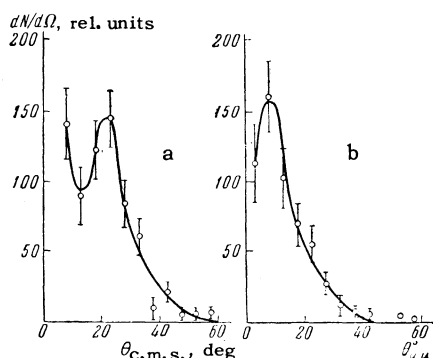


FIG. 2. Angular distributions for Be^8 nuclei in the c.m.s. a) For Be^8 production in the ground state; b) for Be^8 production in excited states (the sum of all excited states). The solid curves were drawn through the experimental points.

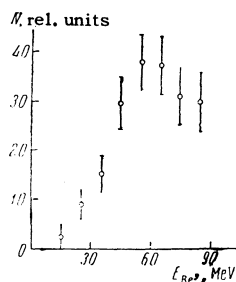


FIG. 4. Yield of Be^8 as function of Be^9 ion energy.

It is of interest to estimate the most probable value of the interaction distance (distance of closest approach of the interacting nuclei $R_{\min}$) at which the neutron transfer occurs. If it is assumed that the bombarding ions follow Rutherford trajectories, then the distance of closest approach of the interacting nuclei is given by the following relation:

$$R_{\min} = \frac{Z_1 Z_2 e^2}{2E} \left(1 + \operatorname{cosec} \frac{\theta_0}{2} \right), \quad (2)$$

where $R_{\min} = r_0 (A_1^{1/3} + A_2^{1/3})$; here $Z_1 e$ and $Z_2 e$ are the charges of the interacting nuclei, A_1 and A_2 are their mass numbers, E is the energy of the bombarding ion, and θ_0 is the angle of emission of the reaction product (Be^8) (all in the c.m.s.).

Histograms of the values of r_0 are shown in Figs. 5 and 6, respectively, for the production of Be^8 in the ground and excited states. The value $r_0 \approx 1.95 \times 10^{-13}$ cm, obtained in the case of Be^8 production in the ground state if allowance is made for the fact that the present data are averaged over two nuclei (Ag and Br), is in rather good agreement with the value $r_0 = 2.15 \times 10^{-13}$ cm obtained by Newman (private communication) for the case of neutron transfer and the production of both final products in the ground state. Some broadening of the histogram toward large values of r_0 can be explained by the existence of two interaction mechanisms. In the grazing interaction, which occurs at smaller collision parameters, the attractive nuclear potential decreases the resulting deflection of the reaction products, which, with the use of relation (2), leads to an apparent increase in the interaction distance. This obviously leads to the broadening of the histogram.

In the case of Be^8 production in excited states, the quantity r_0 cannot be assigned any definite value (Fig. 6). The histogram has, in fact, two quite definite groups of values of r_0 : one in the region $r_0 \sim (1.3-1.6) \times 10^{-13}$ cm and the other in the region $r_0 \sim (2.1-2.3) \times 10^{-13}$ cm. If we start, first, from the assumption that Be^8 is produced in the excited state as a result of grazing interactions, as follows from their angular distribution (Fig. 2b), and, second, that the stronger excitations of the interacting nuclei arise for inter-

FIG. 5. Distribution of r_0 values for the production of Be^8 in the ground state.

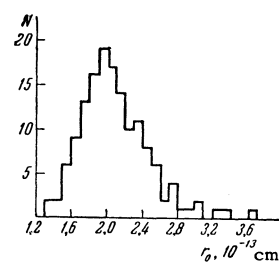
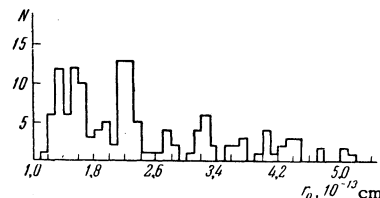


FIG. 6. Distribution of r_0 values for the production of Be^8 in excited states (the sum of all excited states).



actions at smaller distances, then, according to (2), a slight excitation of the final reaction products should lead to a small apparent increase in r_0 as compared to r_0 for the ground state. Consequently, the second group of values of r_0 in Fig. 6 can be explained as being due to the production of the reaction products in states with a small excitation in grazing interactions.

For interactions at smaller distances the attractive nuclear potentials cannot fully offset the Coulomb repulsion of Be^8 , but can cause a still greater deflection (which is not observed in angular distribution in Fig. 2b), which, according to relation (2), should give a value of r_0 smaller than for the ground state.

It is doubtful, however, that relation (2) is applicable in the case of such excitations. Moreover, the cross section of such a process should be extremely small, owing to the strongly competitive absorption process. Hence further investigations in which data for each excited state are obtained are necessary for any definite conclusions.

The author takes this opportunity to express his sincere gratitude to A. P. Klyucharev, N. Ya. Rutkevich, and Yu. N. Ranyuk for very valuable and helpful discussions. The author is also very grateful to L. I. Bolotin for making it possible to carry out this experiment at the heavy-ion accelerator and to Yu. F. Kulygin, G. M. Skoromnyĭ, and O. A. Fedorov, who were responsible for the operation of the accelerator. The author cannot fail to express his gratitude for the great amount of work performed by the microscopist E. V. Cherkavskaya in connection with the analysis of the emulsion.

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PARAMAGNETIC RESONANCE OF SILVER ATOMS TRAPPED IN POLAR MEDIA AT 77°K

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Free silver atoms were trapped in polar media (distilled water, absolute ethanol) at the temperature of liquid nitrogen by the condensation method. The paramagnetic resonance spectra of these atoms were obtained and investigated.

In previous work^[1] free silver atoms were trapped in nonpolar media at the temperature of liquid nitrogen and their paramagnetic resonance spectra investigated. It was of interest to investigate the possibility of trapping the same atoms in polar media and likewise study the paramagnetic resonance of such trapped atoms. This study is the subject of the present paper.

Up to this time the paramagnetic resonance of atoms trapped in polar media has been observed only for hydrogen atoms obtained by irradiation of aqueous solutions of acids frozen at liquid-nitrogen temperatures.^[2,3]

In the present work, water and ethanol were chosen as the polar matrices, and the trapping was accomplished, as in^[1], by the simultaneous condensation of the free atoms and the vapors of the matrix substance on a surface cooled by liquid nitrogen.

EXPERIMENTAL METHOD

As before,^[1] the samples containing the trapped neutral atoms were obtained at the temperature of liquid nitrogen directly in the cavity of the spectrometer used to study their paramagnetic resonance spectra. The apparatus for obtaining the samples is basically the same as that shown in Fig. 1 of^[1]. In distinction to earlier work, in which the substances used for the matrix were solid at room temperature, this time the matrices had to be formed from substances that are liquid at this temperature and required cooling to lower their vapor pressure. This required a number of changes in the apparatus. The vessel containing the matrix substance was removed from the apparatus and placed in a dewar full of alcohol. The alcohol in the dewar was chilled to a low temperature with liquid nitrogen. The temperature of the alcohol bath was measured with a thermocouple.

The vapors of the matrix substance were conducted by a special tube straight to the bottom of the dewar in which the sample was condensed. In addition, in the present work vaporization of the silver with a molybdenum coil was replaced by vaporization from a tungsten boat heated by an electric current.

The same X-band spectrometer with high-frequency modulation used in^[1] was also used for these experiments. The cylindrical transmission-type cavity had 15 mm holes in the end plates, through which passed the quartz tube of the sample-preparation apparatus (see^[1]).

The dc magnetic field was measured to one part in 10⁴ by nuclear magnetic resonance. Field markers were introduced in the recorded traces. The frequency of the klystron was measured with a heterodyne wavemeter with an accuracy of 5 × 10⁻⁵.

EXPERIMENTAL RESULTS

The ground state of the silver atom is ²S_{1/2}. Hence the energies of the Zeeman levels of the hyperfine structure of the ground state of free silver atoms are given by the following Breit-Rabi formula^[4]:

$$W_{I \pm 1/2} = -\frac{\Delta W}{2(2I + 1)} + g_I \beta H m \pm \frac{\Delta W}{2} \left(1 + \frac{4m}{2I + 1} x + x^2 \right)^{1/2}. \quad (1)$$

Here ΔW is the energy of the hyperfine splitting of the ground state in zero magnetic field; I is the nuclear spin; $g_I = -\mu_I/\beta I$ is the nuclear gyromagnetic ratio; μ_I is the nuclear magnetic moment; β is the Bohr magneton; H is the dc magnetic field in which the atom is placed; m is the magnetic quantum number of the total moment of the atom; $\mathbf{F} = \mathbf{J} + \mathbf{I}$, where $\mathbf{J}$ and $\mathbf{I}$ are the moments of the shell and nucleus, respectively; $x = (g_J - g_I) \beta H / \Delta W$ is a dimensionless quantity proportional to the magnetic field H ; g_J is the electronic gyromagnetic ratio of the atom (Landé factor).

In accordance with the considerations presented in [1], the Zeeman levels of atoms trapped in passive media and remaining neutral within them can also be described by Eq. (1), but with different values of ΔW and g_J . Deviations in the values of these quantities from their values for free atoms depend on the interaction of the matrix with the trapped atom. Therefore, having determined from paramagnetic resonance experiments the quantities ΔW and g_J for trapped atoms through the use of Eq. (1) and comparing them with the analogous quantities for free atoms, one can judge the nature of the perturbation of the shell of the trapped atom by the matrix.

Silver has two stable isotopes, Ag^{107} and Ag^{109} , which exist in nature in almost equal proportion. The spins of the nuclei of both isotopes $I = 1/2$, and the nuclear magnetic moments are negative and close in magnitude. Therefore, four Zeeman levels of the hyperfine structure appear in a magnetic field for each isotope; in strong fields it is possible to observe two resonance transitions among them [1]:

$$(F = 1, m = -1) \rightarrow (F = 0, m = 0) \\ \text{and } (F = 1, m = 0) \rightarrow (F = 1, m = 1).$$

Consequently, the paramagnetic resonance spectrum of each silver isotope consists in strong fields of two lines lying on either side of and at equal distances from the line for the free electron at $g = 2.0$. Since the nuclear moments and the quantities ΔW of the silver isotopes differ somewhat, the complete paramagnetic resonance spectrum of the silver atoms will have a characteristic form of four lines, distributed in pairs on either side of the free electron line. [1]

In paramagnetic resonance experiments the resonance frequency ν is usually kept constant while the magnetic field H is varied. Taking this into consideration, we obtain from Eq. (1) the following expressions for the aforementioned two transitions for each silver isotope:

$$\nu = -\Delta\nu \left\{ \frac{1}{2} (1 + x_1^2)^{1/2} + \frac{1}{2} (1 - x_1) - \frac{g_I \beta H_1}{h \Delta\nu} \right\}, \\ \nu = -\Delta\nu \left\{ \frac{1}{2} (1 + x_2^2)^{1/2} - \frac{1}{2} (1 + x_2) - \frac{g_I \beta H_2}{h \Delta\nu} \right\}, \quad (2)$$

where

$$x_1 = (g_J - g_I) \beta H_1 / h \Delta\nu, \quad x_2 = (g_J - g_I) \beta H_2 / h \Delta\nu,$$

H_1 and H_2 are the values of the magnetic field for the first and second transitions; $\Delta\nu = \Delta W/h$, where h is Planck's constant. The nuclear gyromagnetic ratios are $g_I(\text{Ag}^{107}) = 1.237 \times 10^{-4}$ and $g_I(\text{Ag}^{109}) = 1.422 \times 10^{-4}$.

In what follows we shall determine the values of $\Delta\nu$ and g_J from the paramagnetic resonance experiments, using Eq. (2), and then compare them with the analogous values for free atoms.

1. Silver atoms trapped in water at 77°K.

Doubly-distilled water was used as the matrix source in these experiments. To obtain samples with trapped atoms, the ampule containing the water was submerged in the alcohol bath and kept at a temperature of about -60°C . The vapor pressure of water over ice under these conditions is about 8×10^{-3} mm Hg. Water vaporized in vacuum from the solid at this temperature undergoes an additional distillation and becomes extremely pure. Hence the matrix obtained from such vapors was very clean.

Figure 1a shows the paramagnetic resonance spectrum of silver atoms trapped in such a matrix. (All the spectra presented here are differential absorption spectra.)

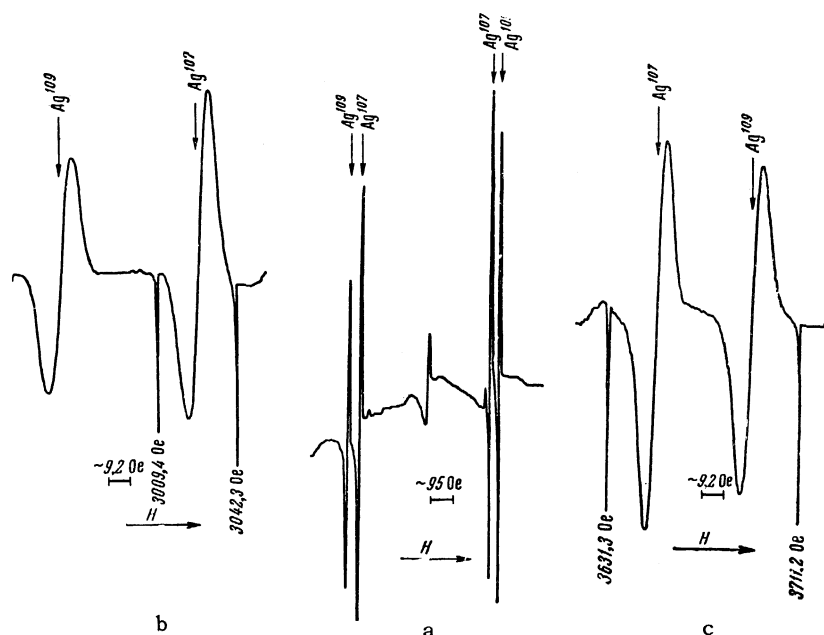
All the experiments described here were carried out at liquid-nitrogen temperature.

The spectrum in Fig. 1a consists of the four lines characteristic of silver atoms. Between the pairs of silver lines there occurs a relatively weak line in the center of the spectrum at $g = 2.0$. This line is due to a free radical created during the capture of the silver atoms in the water. We have not investigated the nature of this radical. It is possibly the OH radical, formed by hydrogen atoms knocked off the water molecules by silver atoms in the condensation process.

Figure 1b shows the low-field pair of lines at slow sweep. The high-field pair is similarly shown in Fig. 1c. The field markers can be seen in these traces. The resonance field values derived from these markers, together with the resonance frequency indicated, suffice to determine the quantities $\Delta\nu$ and g_J for the trapped silver atoms.

These quantities for silver atoms trapped in water are presented in the table. Data such as those shown in Fig. 1 are obtained rather easily, and we repeated these experiments many times. The table gives the mean values obtained from a series of 11 runs. The table also gives the values for free silver atoms. For comparison of these with the values for the trapped atoms, the table gives the quantity $\delta(\Delta\nu)/\Delta\nu$, where $\delta(\Delta\nu) = \Delta\nu - \Delta\nu_{\text{free}}$, expressed as a percentage. The limits of error indicated in the table represent the maximum deviations from the mean for each series of runs.

FIG. 1. Paramagnetic resonance spectrum of silver atoms trapped in H_2O ; the frequency is 9435.8 Mc/sec.



Atom	Matrix	Number of runs	$\Delta\nu$, Mc/sec	$\delta(\Delta\nu)/\Delta\nu$, %	g_J
Ag^{107}	Free atom		1712.56 ± 0.04		2.00224 ± 0.00020
Ag^{109}	Free atom		1976.94 ± 0.04		2.00224 ± 0.00020
Ag^{107}	H_2O	11	1736.9 ± 3.1	+1.42	2.0021 ± 0.0005
Ag^{109}	H_2O	11	2004.7 ± 2.9	+1.40	2.0020 ± 0.0006
Ag^{107}	$\text{C}_2\text{H}_5\text{OH}$	6	1500.3 ± 2.4	-12.39	2.0003 ± 0.0008
Ag^{109}	$\text{C}_2\text{H}_5\text{OH}$	6	1733.0 ± 2.3	-12.36	2.0004 ± 0.0009

The traces shown in Fig. 1 are typical of the experiments on trapped silver atoms in water.

As can be seen from the table, the relative shift in $\Delta\nu$ (i.e., $\delta(\Delta\nu)/\Delta\nu$) for silver atoms in water turns out to be positive and equals $\sim 1.4\%$. Aside from the demonstration that neutral silver atoms can be trapped and stabilized in water at 77°K, this is the most important result of these experiments.

The width of the line, defined as the separation between the maximum values of the first derivative of the absorption line, varied between the limits 6.5 and 10.5 Oe.

2. Silver atoms trapped in ethanol at 77°K. The matrix source was 99.5% absolute ethanol. The source was kept at a temperature of approximately -80°C during sample preparation. Such a low-temperature distillation in vacuum provided an additional effective purification of the ethanol, removing dissolved impurities and possible incidental traces of water, the vapor pressure of which at this temperature is several times lower than that of ethanol.

The paramagnetic resonance of silver atoms trapped in an ethanol matrix is shown in Fig. 2a. The characteristic four-line spectrum of silver

atoms is again seen. Between the pairs, in the center of the spectrum, appears a much weaker line ($g = 2.0$) from a free radical created by the silver atoms in the ethanol. We have not investigated this radical.

Figures 2b and 2c show the left and right pairs of Fig. 2a at slow scan and with field markers.

Figure 2 is typical of the many runs that were made on this system. The mean values of $\Delta\nu$ and g_J for a series of 6 runs are given in the table.

The significant result is the relative shift in $\Delta\nu$, which in distinction to the case of the water matrix is negative and larger in absolute magnitude, -12.4% . This difference can be traced directly in Fig. 2a, in which the splitting in magnetic field between the two lines of each isotope is noticeably less than the analogous splitting in Fig. 1a.

The line widths varied within the limits 8–14 Oe and for the same lines were noticeably greater than in the case of water.

Our method did not permit measurements of the concentration of the trapped neutral atoms. However, the total number of isolated atoms in the matrix was determined from the EPR signal

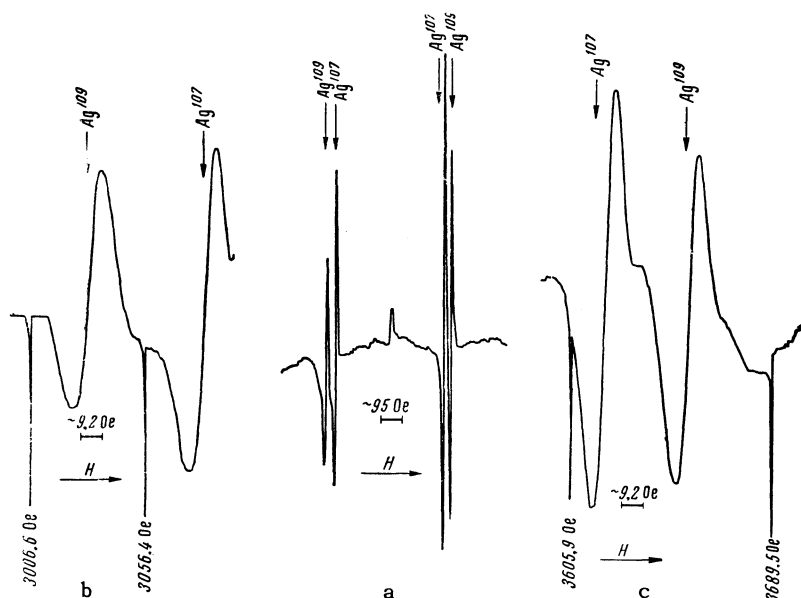


FIG. 2. Paramagnetic resonance spectrum of silver atoms trapped in C_2H_5OH ; the frequency is 9441.2 Mc/sec.

intensity by means of a sample of DPPH. In the majority of the runs with water and ethanol this number exceeded 10^{15} .

The EPR spectra of silver atoms in water and alcohol at liquid-nitrogen temperatures were stable and did not change with time; this is evidence that the trapped atoms are immobile at this temperature.

CONCLUSION

The results presented above demonstrate the possibility of trapping and stabilizing isolated silver atoms in matrices of distilled water and ethanol at 77°K.

The most important results of the experiments with water are the smallness of the shift in $\Delta\nu$ and the positive sign of this shift. The latter is evidence that in this case the Pauli forces^[5] prevail over the electric dipole-dipole forces in the atom-matrix interaction. It is strange that in a matrix of such a polar substance as water, which has a dipole moment of 1.94 D, the shift in $\Delta\nu$ for trapped silver atoms (+1.4%) is almost the same as the shift of +1.15% for silver atoms in paraffin,^[1] which has a molecular dipole moment of zero.

The experiments on trapped silver atoms in ethanol also gave extremely interesting results. In a matrix of ethanol, i.e., of a polar substance whose dipole moment is 1.7 D, the shift turned out to be -12.4%, i.e., not only negative, but also greater in absolute value by about an order of magnitude than in water and paraffin. It can therefore be assumed that in ethyl alcohol the electric dipole-dipole interactions between the trapped atoms and the matrix predominate.^[5]

Thus, in matrices of such similar polar substances as water and ethanol, the shifts in hyperfine splitting for trapped silver atoms were found to be different in sign and completely different in absolute magnitude. On the other hand, for matrices of such different substances as polar water and nonpolar paraffin, these shifts agree in sign and are almost equal in magnitude. At the present time, it is evidently impossible to give an explanation for all these facts since we have as yet no information about the structure of the deposits of water and ethanol condensed in vacuum at 77°K, and consequently nothing is known about the nature of the surroundings of the atoms trapped in such matrices. Besides, no theory has yet been developed to describe the interactions with the matrix and their effect on the EPR spectra of atoms trapped in polar media.

As has been mentioned, the width of the silver lines in alcohol were usually somewhat greater than the width of the same lines in water. Besides, the lines on the low-field side were as a rule somewhat wider than those on the high-field side in the same run.

It should be noted that the matrices were of a high degree of purity. This was achieved by the use of pure starting materials and by the low-temperature distillation in vacuum. The trapping of silver atoms in a matrix obtained by the above-described method from ordinary technical 96% ethanol (rectified) was investigated. In this case a shift of -12.1% (average of three runs) was obtained, i.e., the water impurity in the starting material gave a deviation from the value of -12.4% obtained using absolute alcohol.

The weakness of the free radical signals in Figs. 1a and 2a is evidence that extraneous paramagnetic impurities were present in extremely small amounts in the investigated samples.

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RESTRICTIONS ON THE VALUES OF COUPLING CONSTANTS IN QUANTUM FIELD THEORY. I

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Starting from the Lehmann-Källén expansion for the Green's function and using the dispersion relation for the vertex part $\Gamma(\kappa^2)$, we show that with prescribed masses there must be an upper bound on the coupling constant g^2 . A concrete expression is found for the upper bound on the coupling constant, which shows, for example, that in the case of the deuteron $g^2 < 16m_d^2(\Delta/m)^{1/2}$ (m_d , m are the masses of the deuteron and nucleon, and Δ is the binding energy), and in the case of the interaction between π mesons and nucleons $f^2 < 0.47$.

INTRODUCTION

IN the usual formulation of quantum field theory values are independently prescribed for the masses of the particles and the coupling constants of the various fields. Meanwhile, in a number of papers^[1-5] it has been shown that with given values of the masses there are definite upper bounds on the physical (renormalized) coupling constants, and that these bounds are functions of the masses of the particles. In these papers, however, the treatment either was based on the use of models, or else^[4] depended on the hypothesis that the interaction of elementary particles is characterized by an effective radius which depends only on the masses of the particles and does not increase with an increase of the coupling constant. In the present paper we obtain restrictions on the sizes of the coupling constants for prescribed masses, on the basis of general principles of quantum field theory, without any additional assumptions and without resorting to the use of models.

We base our treatment on the representation of the Green's function in the form of a Lehmann-Källén expansion^[6]; that is to say, we shall assume that all of the conditions for the validity of this expansion are satisfied (existence of a complete system of functions in Hilbert space, definite metric, Lorentz invariance). Starting from the Lehmann-Källén expansion we shall obtain a relation, Eq. (10), which restricts the maximum possible value of the coupling constant g^2 of three fields.

In the boson case this relation involves the vertex part $\Gamma(\kappa^2)$. As a second postulate we assume that the vertex part $\Gamma(\kappa^2)$ is an analytic function

of κ^2 with a cut beginning at the square of the sum of the masses of the nearest two particles, and that a dispersion relation can be written for $\Gamma(\kappa^2)$. The number of subtractions in this dispersion relation is uniquely determined from the same relation, Eq. (10). The maximum value of g^2 will be obtained by minimizing the expression (10) with respect to all possible functions $\Gamma(\kappa^2)$ having the given analytic properties. Variation of the function $\Gamma(\kappa^2)$ leads to an integral equation for $\text{Im } \Gamma(\kappa^2)$, which can be solved and gives restrictions on the coupling constant which depend on the masses of the particles. In the case in which the mass of one of the interacting particles is close to the sum of the masses of the other two, the limit so obtained is twice the limit which follows from non-relativistic quantum mechanics for an interaction with an infinitely small radius.^[2,3,5]

The treatment for the fermion case is similar in principle but is somewhat complicated by the presence of two functions ρ_1 and ρ_2 which are involved in the Lehmann-Källén expansion, and by the fact that the expression for the vertex part is more complicated. Nevertheless, here also we are able, within the framework of the same hypotheses, to get an explicit expression for the limit on the coupling constant. In particular, the coupling constant f^2 for the interaction between π mesons and nucleons must be smaller than 0.47, and if Σ and Λ have opposite parities the constant g^2 for the $\Lambda\Sigma\pi$ interaction must be smaller than 3.2.

BOSONS

The Lehmann-Källén representation for the

Green's function of a boson with spin zero and mass m_a is^[6]:

$$D(\kappa^2) = \frac{1}{\kappa^2 - m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\rho(\kappa'^2)}{\kappa'^2 - \kappa^2 - i0} d\kappa'^2, \quad \rho(\kappa^2) \geq 0, \quad (1)$$

where m_b and m_c are the masses of the nearest (in sum of masses) particles into which particle a can be converted in a transition.

We are interested in the value of the coupling constant g^2 of the three fields a, b, c . To get restrictions on the value of g^2 , we shall first proceed by a method analogous to that of a paper by Lehmann, Symanzik, and Zimmermann.^[7] It can easily be seen from the relation (1) that the function $-D(\kappa^2)$ is an R-function in the complex plane of κ^2 , i.e., that the sign of the imaginary part of $-D(\kappa^2)$ is the same as the sign of the imaginary part of κ^2 . It follows from this that the function $D^{-1}(\kappa^2)$ is also an R-function, and consequently the most general expression for $D^{-1}(\kappa^2)$ is of the form^[1]

$$D^{-1}(\kappa^2) = \frac{1}{\pi} \int_{(m_b+m_c)^2}^{\infty} \text{Im } D^{-1}(\kappa'^2) \left[\frac{1}{\kappa'^2 - \kappa^2} - \frac{1}{\kappa'^2 - m_a^2} \right] d\kappa'^2 + \sum_n R_n \left(\frac{1}{\kappa_n^2 - \kappa^2} - \frac{1}{\kappa_n^2 - m_a^2} \right) + \alpha(\kappa^2 - m_a^2), \quad (2)$$

where the constants R_n , κ_n^2 , and α are real and positive, and

$$m_a^2 < \kappa_n^2 < (m_b + m_c)^2, \quad D^{-1}(m_a^2) = 0,$$

$$\text{Im } D^{-1}(\kappa^2) = \pi \rho(\kappa^2) / |D(\kappa^2)|^2. \quad (2')$$

The relation (2) is written with one subtraction (at the point m_a^2). We shall show that this is sufficient.

It can be seen from the Lehmann-Källén expansion that for large κ^2 the function $D^{-1}(\kappa^2)$ cannot increase more rapidly than κ^2 . Therefore the function $D^{-1}(\kappa^2)/(\kappa^2 - m_a^2)^2$ can be expressed as a Cauchy integral over the usual path:

$$D^{-1}(\kappa^2) = [D^{-1}(m_a^2)]'(\kappa^2 - m_a^2) + \frac{(\kappa^2 - m_a^2)^2}{\pi} \int_{(m_b+m_c)^2}^{\infty} \frac{\text{Im } D^{-1}(\kappa'^2)}{(\kappa'^2 - \kappa^2)(\kappa'^2 - m_a^2)^2} d\kappa'^2 + R(\kappa^2), \quad (3)$$

where $R(\kappa^2)$ is the contribution from the poles of $D^{-1}(\kappa^2)$. It follows from Eq. (3) that the integral

$$\int \frac{\text{Im } D^{-1}(\kappa^2)}{(\kappa^2)^2} d\kappa^2$$

converges, because otherwise for large κ^2 the function $D^{-1}(\kappa^2)$ would increase more rapidly than κ^2 , which is impossible. Thus the integral in the formula (2) converges, and one subtraction is enough.

Let us differentiate Eq. (2) with respect to κ^2 and set $\kappa^2 = m_a^2$. Then, using the equation $[D^{-1}(\kappa^2)]'_{\kappa^2=m_a^2} = 1$, we get

$$\int_{(m_b+m_c)^2}^{\infty} \frac{\rho(\kappa^2)}{|D(\kappa^2)|^2} \frac{d\kappa^2}{(\kappa^2 - m_a^2)^2} + \sum_n R_n \frac{1}{(\kappa_n^2 - m_a^2)^2} + \alpha = 1. \quad (4)$$

Dropping the two positive terms in the left member of Eq. (4), we arrive at the inequality

$$\int_{(m_b+m_c)^2}^{\infty} \frac{\rho(\kappa^2)}{|D(\kappa^2)|^2} \frac{1}{(\kappa^2 - m_a^2)^2} d\kappa^2 \leq 1. \quad (5)$$

As is shown in Lehmann's paper,^[6] $\rho(\kappa^2)$ can be written in the form

$$\rho(\kappa^2) = (2\pi)^3 \sum_n |a_{0n}|^2, \quad (6)$$

where

$$a_{0n} = \langle 0 | A(0) | \Phi_n \rangle, \quad (6')$$

$A(x)$ is the field of particle a , and Φ_n is an arbitrary state of the complete system of functions with four-momentum k such that $k^2 = \kappa^2$. It is clear from Eq. (6) that the inequality (5) is only strengthened if in the summation over n in Eq. (6) we take into account only two-particle states of particles b and c .

The contribution to $\rho(\kappa^2)$ owing to the two-particle states can be expressed in terms of the Green's function $D(\kappa^2)$ and the vertex part $\Gamma(\kappa^2)$. We shall suppose that particles b and c are scalar particles and that the product of the parities of all three particles a, b , and c is unity. Then from the definition of the vertex part

$$g\Gamma(x, y, z)$$

$$= \int d^4x' d^4y' d^4z' D_a^{-1}(z - z') D_b^{-1}(x - x') D_c^{-1}(y - y') \times \langle 0 | T \{ A(z'), \Phi_b(x'), \Phi_c(y') \} | 0 \rangle \quad (7)$$

we can obtain a connection between the matrix element $\langle 0 | A(0) | \Phi_{k_b, k_c} \rangle$ (where Φ_{k_b, k_c} is the two-particle state of particles b and c with the momenta k_b, k_c , and $(k_b + k_c)^2 = \kappa^2$, $k_b^2 = m_b^2$, $k_c^2 = m_c^2$) and the vertex part $\Gamma(m_b^2, m_c^2, \kappa^2) \equiv \Gamma(\kappa^2)$:

$$\langle 0 | A(0) | \Phi_{k_b, k_c} \rangle = g D_a^{-1}(\kappa^2) \Gamma(\kappa^2) / \sqrt{4\pi\omega_b\omega_c} \quad (8)$$

(ω_b and ω_c are the energies of particles b and c). Substituting Eq. (8) in Eq. (6), we find the contribution to $\rho(\kappa^2)$ owing to two-particle states:

$$\rho_{\text{two-particle}} = (1/2\pi) g^2 |D(\kappa^2)|^2 |\Gamma(\kappa^2)|^2 q(\kappa^2) / \kappa, \quad (9)$$

$$q(\kappa^2) = \sqrt{[\kappa^2 - (m_b + m_c)^2][\kappa^2 - (m_b - m_c)^2]} / 2\kappa. \quad (9')$$

After substituting Eq. (9) in Eq. (5) we arrive at an inequality which restricts the possible values

of the renormalized coupling constant g^2 of the three fields a, b, c ¹⁾:

$$\frac{g^2}{2\pi} \int_{(m_b+m_c)^2}^{\infty} \frac{|\Gamma(\kappa^2)|^2}{(\kappa^2 - m_a^2)^2} \frac{q(\kappa^2)}{\kappa} d\kappa^2 < 1. \quad (10)$$

In order to get from the inequality (10) definite restrictions on g^2 for prescribed masses m_a, m_b, m_c it is necessary that the value of the integral

$$\Phi = 2(m_b + m_c)^2 \int_{(m_b+m_c)^2}^{\infty} \frac{|\Gamma(\kappa^2)|^2}{(\kappa^2 - m_a^2)^2} \frac{q(\kappa^2)}{\kappa} d\kappa^2 \quad (11)$$

have a lower bound.

We shall look for the minimum of Φ over the class of functions $\Gamma(\kappa^2)$ that have the following properties:

1) $\Gamma(\kappa^2)$ is a holomorphic function of κ^2 in the complex plane of κ^2 with a cut along the real axis which begins at the point $\kappa^2 = (m_b + m_c)^2$.²⁾ On the real axis to the left of the point $\kappa^2 = (m_b + m_c)^2$ the function $\Gamma(\kappa^2)$ is real;

2) the increase of $\Gamma(\kappa^2)$ at infinity is not faster than a power law;

3) the value at the point $\kappa^2 = m_a^2$ is $\Gamma(m_a^2) = 1$.

We assume that $\Gamma(\kappa^2)$ has no poles in the complex plane of κ^2 .³⁾ In principle there could be poles of $\Gamma(\kappa^2)$ on the real axis in the interval $m_a^2 < \kappa^2 < (m_b + m_c)^2$, at the points κ_n^2 at which the Green's function $D(\kappa^2)$ is equal to zero, while $\Gamma(\kappa^2)D(\kappa^2) \rightarrow \text{const}$ for $\kappa^2 \rightarrow \kappa_n^2$. [The last fact follows, for example, from the Schwinger equation for the Green's function $D(\kappa^2)$]. These poles of $\Gamma(\kappa^2)$ correspond to true bound states of the particles b and c , and owing to the terms $g^2\Gamma^2(\kappa^2)D(\kappa^2)$ they will lead to poles in the scattering amplitude of particles b and c . [This situation occurs, for example, in the theory of superconductivity, where a pole of $\Gamma(\kappa^2)$ corresponds to a bound state of a Cooper pair.] Thus our assumption means that there are no particles with the same quantum numbers as particle a and lying between m_a and $m_b + m_c$.

The function $\Gamma(\kappa^2)$ will automatically have the properties 1)–3) if we write the dispersion relation

$$\Gamma(\kappa^2) = 1 + \frac{\kappa^2 - m_a^2}{\pi} \int_{(m_b+m_c)^2}^{\infty} d\kappa'^2 \frac{\text{Im } \Gamma(\kappa'^2)}{(\kappa'^2 - \kappa^2 - i\delta)(\kappa'^2 - m_a^2)}, \quad (12)$$

with an arbitrary function $\text{Im } \Gamma(\kappa^2)$ with one subtraction. It follows from the inequality (10) that on the real axis $|\Gamma(\kappa^2)|$ increases more slowly than $(\kappa^2)^{1/2}$. By using the condition (2) we can show from this that one subtraction is surely enough in the dispersion relation for Γ .

We can now find the minimum of the functional $\Phi[\Gamma(\kappa^2)]$ for the class of functions $\Gamma(\kappa^2)$ with which we are concerned. To do so we substitute Eq. (12) in Eq. (11) and perform the integration over κ^2 . After changing to the dimensionless variable $x = \kappa^2/(m_b + m_c)^2$ we get

$$\begin{aligned} \Phi = & \frac{1}{\alpha^2} \left[-\alpha + l + \frac{1}{2} \frac{\alpha(1+\lambda) - 2\lambda}{(1-\alpha)(\alpha-\lambda)} \beta \right] \\ & - \frac{2}{\pi} \int_1^{\infty} \left[\frac{l}{\alpha x} + \frac{\beta}{\alpha(x-\alpha)} - \frac{L(x)}{x(x-\alpha)} \right] \varphi(x) dx \\ & + 2 \int_1^{\infty} \frac{V(x-1)(x-\lambda)}{x} \varphi^2(x) dx \\ & + \frac{1}{\pi^2} \int_1^{\infty} dx \int_1^{\infty} dy \left\{ \frac{l}{xy} + \frac{1}{y-x} \left[\frac{L(x)}{x} - \frac{L(y)}{y} \right] \right\} \varphi(x) \varphi(y), \quad (13) \end{aligned}$$

where

$$\alpha = \frac{m_a^2}{(m_b + m_c)^2}, \quad \lambda = \frac{(m_b - m_c)^2}{(m_b + m_c)^2},$$

$$\varphi(x) = \frac{\text{Im } \Gamma(x)}{x - \alpha}, \quad l = \sqrt{\lambda} \ln \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}},$$

$$\beta = \sqrt{(1-\alpha)(\alpha-\lambda)} \left[\frac{\pi}{2} + \arcsin \frac{2\alpha-1-\lambda}{1-\lambda} \right],$$

$$L(x) = \sqrt{(x-1)(x-\lambda)} \ln \frac{(\sqrt{x-\lambda} + \sqrt{x-1})^2}{1-\lambda}. \quad (13')$$

[The formula (13) is written for the case $\alpha \geq \lambda$, i.e., $m_a \geq |m_b - m_c|$.] Finding the variation of Φ with respect to φ , we get the equation satisfied by the function $\varphi(x)$ which minimizes the integral Φ :

$$\begin{aligned} & \frac{V(x-1)(x-\lambda)}{x} \varphi(x) + \frac{1}{2\pi^2} \int_1^{\infty} \left\{ \frac{l}{xy} + \frac{1}{y-x} \left[\frac{L(x)}{x} - \frac{L(y)}{y} \right] \right\} \varphi(y) dy \\ & - \frac{1}{2\pi} \left[\frac{l}{\alpha x} + \frac{\beta}{\alpha(x-\alpha)} - \frac{L(x)}{x(x-\alpha)} \right] = 0. \quad (14) \end{aligned}$$

When we use Eq. (14) the minimum value of Φ can be written

$$\begin{aligned} \Phi_{\min} = & \frac{1}{\alpha^2} \left[-\alpha + l + \frac{1}{2} \frac{\alpha(1+\lambda) - 2\lambda}{(1-\alpha)(\alpha-\lambda)} \beta \right] \\ & - \frac{1}{\pi} \int_1^{\infty} dx \varphi(x) \left[\frac{l}{\alpha x} + \frac{\beta}{\alpha(x-\alpha)} - \frac{L(x)}{x(x-\alpha)} \right]. \quad (15) \end{aligned}$$

¹⁾We use the Gaussian system of units, so that, for example, in our notation $e^2 = 1/137$.

²⁾In the case considered (with b and c the lightest possible particles) there can be no anomalous thresholds in $\Gamma(\kappa^2)$.

³⁾The fact that $\Gamma(\kappa^2)$ can have poles on the real axis in the complex plane of κ^2 has been called to our attention by V. Ya. Fainberg and E. S. Fradkin, to whom we express our deep gratitude.

Equation (14) is a Fredholm integral equation of the second kind, which, as is well known, has a unique solution (provided the parameter $1/2\pi^2$ is not equal to an eigenvalue of the homogeneous equation, and in our case there is no reason for this to be so). In the general case the equation (14) can be solved numerically, but some properties of the solution [and consequently of the $\Gamma(\kappa^2)$ which gives the minimum value of Φ] can be seen directly from the equation.

For x close to unity and $x > 1$, i.e., near the threshold for production of particles b and c , it is not hard to see that $\varphi(x)$ is proportional to $(x-1)^{-1/2}$, so that $\Gamma(x)$ is of the form

$$\Gamma(x) = iA/\sqrt{x-1} + B, \quad (16)$$

where A and B are constants and A is real. Using Eqs. (2), (2'), and (9), one can easily show that if $\Gamma(x)$ near the threshold is given by Eq. (16) and goes to infinity for $x \rightarrow 1$, then the Green's function $D(x)$ will go to zero for $x \rightarrow 1$ and near $x = 1$ will be given by

$$D(x) = \left[i \frac{g^2}{4\pi} A^2 \frac{\sqrt{1-\lambda}}{\sqrt{x-1}} + C \right]^{-1}, \quad (17)$$

where C is a constant. It is interesting to see whether such forms of Γ and D may not lead to an unphysical behavior of the scattering amplitude of particles b and c near the threshold. To calculate the s -wave scattering amplitude $a_0(x)$ for particles b and c we can use the relation

$$\text{Im} [\Gamma(x) D(x)] = \Gamma(x) D(x) a_0^*(x) q(x). \quad (18)$$

Substituting Eqs. (16) and (17) in Eq. (18), we find that for $x \rightarrow 1$

$$a_0 q = e^{i\delta} \sin \delta = \gamma \sqrt{x-1} \equiv \gamma' q \quad (19)$$

(γ, γ' are real), i.e., we get the effective-range approximation for the s -wave scattering amplitude.

It must be noted that although when we calculate the scattering amplitude of particles b and c near threshold in this way we get a quite reasonable expression, we nevertheless cannot be sure that the $\Gamma(x)$ determined from Eq. (14) can correspond to an actual physical situation. In fact, if we use the $\Gamma(x)$ of Eq. (15) and the $D(x)$ of Eq. (16) to calculate the pole term in the scattering amplitude $a_0(x)$ of particles b and c near threshold, $(g^2/2\pi) \Gamma^2(x) D(x)$, it will be proportional to $(x-1)^{-1/2}$, and consequently in order for the effective-range approximation to hold it is necessary that the pole term be compensated by some sort of

more complicated diagrams, and this seems very strange.⁴⁾

Thus when we find the minimum of the functional Φ by means of the function $\varphi(x)$ determined from Eq. (14) we may possibly be getting too low a value for this minimum (and consequently too high a limit on g^2), in comparison with what can in principle be realized in actual physical problems.

Let us now consider the limiting case of small values of $1-\alpha \ll 1$, i.e., the case $\Delta = m_b + m_c - m_a \ll m_a$. This case corresponds to the non-relativistic approximation; particle a can be interpreted as a bound state of particles b and c , and Δ is the binding energy.

It is not hard to see that when $1-\alpha$ is small the values of x that are important for the integral in Eq. (15) are those close to unity, $x-1 \sim 1-\alpha$. Then we can neglect the integral term in Eq. (14), and⁵⁾

$$\varphi(x) = \sqrt{1-\alpha}/2 (x-\alpha) \sqrt{x-1}. \quad (20)$$

For $\Phi_{\min}$ we get

$$\Phi_{\min} = \frac{\pi}{4} \sqrt{\frac{1-\lambda}{1-\alpha}} = \frac{\pi}{4} \sqrt{\frac{2\mu}{\Delta}}, \quad (21)$$

where $\mu = m_b m_c / (m_b + m_c)$ is the reduced mass of particles b and c . For $\Gamma(x) \equiv 1$ we would have had $\Phi = (\pi/2)(2\mu/\Delta)^{1/2}$, i.e., the value of the integral Φ cannot be decreased by more than a factor two because $\Gamma(x)$ is different from unity. We get as the result the following upper limit on the coupling constant g^2 :

$$g^2 < 16m_a^2 \sqrt{\Delta/2\mu}. \quad (22)$$

A physical example of such a situation is the case of the deuteron; the deuteron is particle a , and particles b and c are the neutron and proton.

⁴⁾If there were no such cancellation it would mean that the scattering amplitude has a resonance at zero energy.

⁵⁾It can easily be shown from Eq. (14) that in the nonrelativistic case the vertex part Γ that gives the minimum of Φ and the corresponding Green's function D are

$$\Gamma(\epsilon) = 1/2 (1 + i \sqrt{\Delta/\epsilon}), \quad D(\epsilon) = -i (8m_a/g^2 \sqrt{2\mu}) \sqrt{\epsilon}/(\epsilon + \Delta),$$

where ϵ is the kinetic energy of particles b and c ; $\kappa = m_b + m_c + \epsilon$. Calculating the scattering amplitude according to Eq. (18), we find $a_0 = i[(2\mu\epsilon)^{1/2} - i(2\mu\Delta)^{1/2}]^{-1}$ — the usual expression for the scattering amplitude when there is a bound state in a potential with an infinitely small range (the Bethe-Peierls formula for np scattering in the triplet state). It is important to point out that the usual expression for the scattering amplitude has been obtained here with $\Gamma(\epsilon) = 1/2[1 + i(\Delta/\epsilon)^{1/2}]$, and not with $\Gamma(\epsilon) = 1$, as in the Bethe-Peierls theory. For this reason the bound on g^2 was twice the usual value.

The fact that the deuteron, neutron, and proton have spins is unimportant, since at small energies the interaction is that in the triplet state and the spin functions can be factored out.

The fact that at small values of the binding energy Δ the coupling constant g^2 decreases in proportion to $\Delta^{1/2}$ has been derived in a number of papers,^[2,3,5] which treated the problem on the basis of nonrelativistic theory and with the range of the nuclear forces set equal to zero. With these assumptions the limit obtained on g^2 is smaller than ours by a factor two (i.e., is the same as would be obtained with $\Gamma = 1$). In the paper of Gribov, Zel'dovich, and Perelomov^[3] it was pointed out that inclusion of a finite range of the nuclear forces raises the limit on the coupling constant g^2 . It follows from our treatment that the increase of the value of g^2 due to the finite range of nuclear forces cannot be by more than a factor two. From the experimental data on the scattering of neutrons by protons (cf. e.g.,^[8]) one gets

$$g^2 = 12m_D^2 \sqrt{\Delta/2\mu}.$$

These data show that for the case of the deuteron the estimate (22) is a very satisfactory one.

The numerical solution of Eq. (14) and the determination of $\text{Im } \Gamma(x)$ and $\Phi_{\min}$ from Eq. (15) have been carried out for a number of values of α and λ by A. S. Kronrod, G. M. Adel'son-Vel'skiĭ, F. M. Filler, and L. V. Il'kov with an electronic computing machine.

The minimum value of Φ can also be obtained analytically by an elegant method^[9-11] pointed out to us by N. N. Meĭman. To find the minimum of the integral

$$\Phi = \int_1^\infty \frac{\sqrt{(x-1)(x-\lambda)}}{x(x-\alpha)^2} |\Gamma(x)|^2 dx \quad (23)$$

in the complex plane of x , one makes the conformal transformation

$$z = -(\sqrt{x-1} - i\sqrt{1-\alpha})/(\sqrt{x-1} + i\sqrt{1-\alpha}), \quad (24)$$

which takes the two sides of the cut along the real axis from unity to infinity over into the unit circle. Then the entire cut plane of x goes over into the interior of the unit circle, and the point x into the center of the circle. The integral (23) is transformed into

$$\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) |\Gamma(z)|^2 d\theta, \quad z = e^{i\theta}, \quad (25)$$

$$f(\theta) = \frac{\pi}{\sqrt{1-\alpha}} \frac{u \sqrt{1-\lambda+(1-\alpha)u}}{[1+(1-\alpha)u](1+u)}, \quad u = \tan^2 \frac{\theta}{2}. \quad (26)$$

$\Gamma(z)$ is an analytic function inside the unit circle and $\Gamma(0) = 1$. The solution of the problem of finding the minimum of the integral (25) over the class of functions $\Gamma(z)$ that are analytic inside the circle and have $\Gamma(0) = 1$ is obtained (cf. ^[9-11]) by expanding $\Gamma(z)$ in terms of the system of polynomials orthogonal on the unit circle with the weight $f(\theta)$, and the answer is

$$\Phi_{\min} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(\theta) d\theta \right\}. \quad (27)$$

After an elementary integration we get

$$\Phi_{\min} = \frac{\pi}{4} \frac{\sqrt{1-\lambda} + \sqrt{1-\alpha}}{\sqrt{1-\alpha}(1 + \sqrt{1-\alpha})^2}. \quad (28)$$

[The results of calculation with the formula (28) naturally agree with the results of numerical computation with Eqs. (14) and (15)].

Thus in the case of the interaction of three boson fields the coupling constant g^2 is bounded by the inequality

$$g^2 < 16 \frac{\sqrt{(m_b+m_c)^2 - m_a^2} (m_b+m_c + \sqrt{(m_b+m_c)^2 - m_a^2})^2}{2 \sqrt{m_b m_c} + \sqrt{(m_b+m_c)^2 - m_a^2}}. \quad (29)$$

It is obvious that in the nonrelativistic case ($m_b + m_c - m_a \ll m_a$) the inequality (29) goes over into (22).

FERMIONS

If a particle a is a fermion, the Lehmann-Källén representation for its Green's function is of the form

$$G(\hat{p}) = \frac{1}{\hat{p} - m_a} - \int_{(m_b+m_c)^2}^{\infty} \frac{(\hat{p} + \kappa) \rho_1(\kappa^2) - \rho_2(\kappa^2)}{\kappa^2 - p^2 - i\delta} d\kappa^2, \quad (30)$$

where the functions ρ_1 and ρ_2 are positive and $2\kappa\rho_1 - \rho_2 \geq 0$. Writing $G(\hat{p})$ in the form

$$G(\hat{p}) = \hat{p} f_1(p^2) + f_2(p^2), \quad (31)$$

we get for the functions $f_1(p^2)$ and $f_2(p^2)$ the relations

$$f_1(p^2) = \frac{1}{p^2 - m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\rho_1(\kappa^2)}{\kappa^2 - p^2 - i\delta} d\kappa^2, \quad (32)$$

$$f_2(p^2) = \frac{m_a}{p^2 - m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\kappa \rho_1(\kappa^2) - \rho_2(\kappa^2)}{\kappa^2 - p^2 - i\delta} d\kappa^2. \quad (33)$$

It follows from Eq. (32) that $f_1^{-1}(p^2)$ is an R-function. Then by arguments analogous to those used for the boson case in going from Eq. (1) to Eq. (5), we get

$$\int_{(m_b+m_c)^2}^{\infty} \frac{\rho_1(\kappa^2)}{|\tilde{f}_1(\kappa^2)|^2} \frac{1}{(\kappa^2 - m_a^2)^2} d\kappa^2 \leq 1. \quad (34)$$

Just as before, the inequality (34) is only strengthened if in $\rho_1(\kappa^2)$ we confine ourselves to including only the lowest two-particle states. We shall suppose that the lowest state is a boson with spin zero and a fermion with spin $1/2$. Then^[6]

$$\rho_{1 \text{ two-particle}}(p^2) = \frac{1}{4p_0} (2\pi)^3 \sum_{n,r} C_a^{(r)*} C_a^{(r)}, \quad (35)$$

where

$$C_a^{(r)} = \langle 0 | \psi_a(0) | \Phi_{p_1, r; k} \rangle \quad (36)$$

and $\Phi_{p_1, r; k}$ is the two-particle state of a fermion with momentum p_1 and polarization r and a boson with momentum k , $(p_1 + k)^2 = p^2$. From the definition of the vertex part $\Gamma(x, y; z)$,

$$g\Gamma(x, y; z) = \int d^4x' d^4y' d^4z' G_a^{-1}(x - x') D_c^{-1}(z - z') \times \langle 0 | T \{ \psi(z'), \psi(x'), \bar{\psi}(y') \} | 0 \rangle G_b^{-1}(y - y'), \quad (37)$$

it is easy to show, when we go over to the momentum representation, that $C_a^{(r)}$ is proportional to

$$g \frac{1}{\sqrt{k_0 p_0}} G_a(\hat{p}) \Gamma(\hat{p}) u_{p_1}^r, \quad (38)$$

where $u_{p_1}^r$ is a spinor which describes the free state of fermion b with four-momentum p_1 and polarization r . Substituting this expression for $C_a^{(r)}$ in Eq. (35), we arrive at the following formula for $\rho_{1 \text{ two-particle}} [q = q(p^2)]$ is determined according to Eq. (9')]:

$4p_0 \rho_{1 \text{ two-particle}}$

$$= \frac{1}{2\pi} g^2 \frac{q(p^2)}{\sqrt{p^2}} \text{Sp} \{ \beta \Gamma^+(\hat{p}) G^+(\hat{p}) G(\hat{p}) \Gamma(\hat{p}) (\hat{p}_1 + m_b) \}. \quad (39)$$

Let us consider two cases: a) the scalar case, in which the parity of the boson c is the same as the product of the parities of the fermions a and b ; b) the pseudoscalar case, in which the parity of the boson c is opposite to the product of the parities of the fermions a and b . In the scalar case the general expression for $\Gamma(\hat{p})$ is of the form

$$\Gamma_S(\hat{p}) = \Gamma_{1S}(p^2) + \hat{p} \Gamma_{2S}(p^2), \quad (40)$$

and in the pseudoscalar case

$$\Gamma_{PS}(\hat{p}) = \gamma_5 \Gamma_{1PS}(p^2) + \hat{p} \gamma_5 \Gamma_{2PS}(p^2). \quad (41)$$

[Particle b is a real particle, and therefore for it $\hat{p}_1 = m_b$, so that according to Eq. (38) the only spinor terms remaining in the expression for Γ are those indicated in Eqs. (40), (41).] Substituting Eqs. (31), (40), (41) in Eq. (39) and calculating the trace, we get the following expression for $\rho_{1 \text{ two-particle}}$:

$\rho_{1 \text{ two-particle}}(p^2)$

$$= (g^2/4\pi) \{ |f_1 p + f_2|^2 |\Gamma_1 + p \Gamma_2|^2 [(p \pm m_b)^2 - m_c^2] + |f_1 p - f_2|^2 |\Gamma_1 - p \Gamma_2|^2 [(p \mp m_b)^2 - m_c^2] \} q(p^2)/2p^3, \quad (42)$$

where $p = (p^2)^{1/2}$, and the upper sign holds for the scalar case and the lower for the pseudoscalar case. Substituting Eq. (42) in Eq. (34) and introducing the notations

$$(f_1 \Gamma_1 + f_2 \Gamma_2)/2f_1 = F_1(p^2), (f_1 \Gamma_2 p^2 + f_2 \Gamma_1)/2f_1 m_a = F_2(p^2), \quad (43)$$

we can write the inequality (34) in the form

$$(g^2/2\pi) \Phi < 1, \quad (44)$$

$$\Phi = \int_{(m_b+m_c)^2}^{\infty} \frac{d\kappa^2}{\kappa^2 (\kappa^2 - m_a^2)^2} \frac{q(\kappa^2)}{\kappa} \times \{ |F_1(\kappa^2) \kappa + F_2(\kappa^2) m_a|^2 [(\kappa \pm m_b)^2 - m_c^2] + |F_1(\kappa^2) \kappa - F_2(\kappa^2) m_a|^2 [(\kappa \mp m_b)^2 - m_c^2] \}. \quad (45)$$

Just as in the boson case, we assume that $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ are analytic functions of κ^2 in the entire complex plane of κ^2 with a cut along the real axis from $\kappa^2 = (m_b + m_c)^2$ to infinity, which are real on the real axis to the left of the point $\kappa^2 = (m_b + m_c)^2$ and do not increase at infinity more rapidly than a power of the argument. According to the definition of physical charge and the renormalized vertex part the following condition must hold:

$$\Gamma_1(m_a^2) + m_a \Gamma_2(m_a^2) = 1. \quad (46)$$

In analogy with the boson case we assume that the functions $\Gamma_1(\kappa^2)$ and $\Gamma_2(\kappa^2)$ have no poles anywhere in the complex plane, including the interval from m_a^2 to $(m_b + m_c)^2$ on the real axis. This means that the Green's function of the fermion a has no zeroes associated with poles of Γ_1 and Γ_2 . In addition, we here make the stronger assumption that the Green's function $f_1(p^2)$ of the fermion a has no zeroes at all.⁶⁾

On these assumptions the functions $F_1(\kappa^2)$ and $F_2(\kappa^2)$ are analytic functions of κ^2 in the plane of κ^2 cut along the real axis from $\kappa^2 = (m_b + m_c)^2$ to infinity. On the real axis to the left of $(m_b + m_c)^2$ the functions $F_1(\kappa^2)$ and $F_2(\kappa^2)$ are real, and at the point $\kappa^2 = m_a^2$ they satisfy the conditions

⁶⁾It follows from Eq. (32) that if $f_1(\kappa^2)$ has only one pole, at the point $\kappa = m_a^2$, then in the most general case $f_1(\kappa^2)$ can have not more than one zero, and this zero must lie on the real axis in the interval $m_a^2 < \kappa^2 < (m_b + m_c)^2$. The restrictions on g^2 in the case in which $f_1(\kappa^2)$ has one zero will be dealt with in a later paper.

$$F_1(m_a^2) = F_2(m_a^2) = 1/2. \quad (47)$$

It will be convenient for us to transform the expression (45) for Φ , introducing instead of F_1 and F_2 the new functions $\bar{F}_1(\kappa^2)$ and $\bar{F}_2(\kappa^2)$:

$$F_1(\kappa^2) = \frac{2m_b(m_b + m_c)}{\kappa^2 + m_b^2 - m_c^2} [\mp \bar{F}_1(\kappa^2) + \bar{F}_2(\kappa^2)],$$

$$F_2(\kappa^2) = \frac{m_b + m_c}{m_a} \bar{F}_1(\kappa^2). \quad (48)$$

In terms of the new functions the formula for Φ is

$$\Phi = 8 \int_{(m_b + m_c)^2}^{\infty} \frac{d\kappa^2}{(\kappa^2 + m_b^2 - m_c^2)} \frac{(m_b + m_c)^2}{(\kappa^2 - m_a^2)^2} \frac{q(\kappa^2)}{\kappa} \times [q^2(\kappa^2) |\bar{F}_1(\kappa^2)|^2 + m_b^2 |\bar{F}_2(\kappa^2)|^2]. \quad (49)$$

It is obvious that the functions $\bar{F}_1(\kappa^2)$ and $\bar{F}_2(\kappa^2)$ have the same analyticity properties as F_1 and F_2 . By the same arguments as used in the proof of Eq. (2) one can show that the behavior of $\bar{F}_1$ and $\bar{F}_2$ at infinity is such that for each of them there is a dispersion relation with one subtraction. At the point $\kappa^2 = m_a^2$

$$\bar{F}(m_a^2) = \frac{1}{2} \frac{m_a}{m_b + m_c},$$

$$\bar{F}_2(m_a^2) = \frac{1}{2} \left[\frac{m_a^2 + m_b^2 - m_c^2}{2m_b(m_b + m_c)} \pm \frac{m_a}{m_b + m_c} \right]. \quad (50)$$

From the requirement that $F_1(\kappa^2)$ be regular at the point $\kappa^2 = -m_b^2 + m_c^2$ there follows one further condition on the functions $\bar{F}_1$ and $\bar{F}_2$:

$$\mp \bar{F}_1(-m_b^2 + m_c^2) + \bar{F}_2(-m_b^2 + m_c^2) = 0. \quad (51)$$

Thus in order to determine the maximum possible value of g^2 we must find the minimum of the function Φ over the class of functions $\bar{F}_1$ and $\bar{F}_2$ having the analytic properties indicated above and satisfying the conditions (50) and (51). To solve this problem we substitute the dispersion relations for $\bar{F}_1(\kappa^2)$ and $\bar{F}_2(\kappa^2)$ [with one subtraction at the point $\kappa^2 = m_a^2$] in Eq. (49) and carry out the integration over κ^2 . The result is

$$\Phi = C + \frac{1}{\pi} \int_1^{\infty} dx \{ R_1(x) \varphi_1(x) \sqrt{\alpha} + R_2(x) [\sqrt{\alpha}(1 + \sqrt{\lambda}) \varphi_1(x) - (1 \pm \sqrt{\alpha})(\sqrt{\lambda} \pm \sqrt{\alpha}) \varphi_2(x)] (1 + \sqrt{\lambda}) \} + 2 \int_1^{\infty} dx \frac{\sqrt{(x-1)(x-\lambda)}}{x(x + \sqrt{\lambda})} \times \left[\frac{(x-1)(x-\lambda)}{x} \varphi_1^2(x) + (1 + \sqrt{\lambda})^2 \varphi_2^2(x) \right] + \frac{1}{\pi^2} \int_1^{\infty} dx \int_1^{\infty} dy \{ K_1(x, y) \varphi_1(x) \varphi_1(y) + K_2(x, y) [\varphi_2(x) \varphi_2(y) \varphi_1(x) \varphi_1(y)] (1 + \sqrt{\lambda})^2 \}. \quad (52)$$

Here (we suppose $m_b \geq m_c$)

$$C = \frac{1}{4\alpha^2} \left\{ -\alpha(\alpha + 2\sqrt{\lambda}) + \left[2\lambda - \frac{\alpha}{2}(1 - \sqrt{\lambda})^2 \right] \frac{l}{\sqrt{\lambda}} + \beta \left[2\sqrt{\lambda} + \alpha \frac{(\alpha - \sqrt{\lambda})(1 + \sqrt{\lambda})^2}{2(1 - \alpha)(\alpha - \lambda)} \right] \right\} + \frac{1}{4} \frac{1}{\alpha + \sqrt{\lambda}} [(1 \pm \sqrt{\alpha})^2 (\sqrt{\lambda} \pm \sqrt{\alpha})^2 - \alpha(1 + \sqrt{\lambda})^2] \times \left\{ -\frac{1}{\alpha} + \left[\frac{2\alpha + \sqrt{\lambda}}{\alpha^2(\alpha + \sqrt{\lambda})} + \frac{2\alpha - 1 - \lambda}{2\alpha(1 - \alpha)(\alpha - \lambda)} \right] \beta + \frac{\alpha + \sqrt{\lambda}}{\alpha^2 \sqrt{\lambda}} l - \frac{1 + \sqrt{\lambda}}{\alpha + \sqrt{\lambda}} \frac{1}{\lambda^{1/4}} \ln \frac{1 + \lambda^{1/4}}{1 - \lambda^{1/4}} \right\},$$

$$R_1(x) = \frac{\sqrt{\lambda}}{\alpha x} + \frac{1}{\alpha x} \left[-\frac{x + \alpha}{\alpha x} \lambda + \frac{1}{2}(1 - \sqrt{\lambda})^2 \right] \frac{1}{\sqrt{\lambda}} l - \frac{\alpha + \sqrt{\lambda}}{\alpha^2(x - \alpha)} \beta + \frac{x + \sqrt{\lambda}}{x^2(x - \alpha)} L(x),$$

$$R_2(x) = \frac{1}{\alpha + \sqrt{\lambda}} \left[\frac{\alpha + \sqrt{\lambda}}{\alpha x \sqrt{\lambda}} l + \frac{1}{\alpha(x - \alpha)} \beta - \frac{1 + \sqrt{\lambda}}{x + \sqrt{\lambda}} \frac{1}{\lambda^{1/4}} \ln \frac{1 + \lambda^{1/4}}{1 - \lambda^{1/4}} - \frac{(\alpha + \sqrt{\lambda}) L(x)}{(x + \sqrt{\lambda})x(x - \alpha)} \right],$$

$$K_1(x, y) = -\frac{\sqrt{\lambda}}{xy} + \frac{1}{xy} \left[\frac{x + y}{xy} \lambda - \frac{1}{2}(1 - \sqrt{\lambda})^2 \right] \frac{1}{\sqrt{\lambda}} l - \frac{1}{x - y} \left[\frac{x + \sqrt{\lambda}}{x^2} L(x) - \frac{y + \sqrt{\lambda}}{y^2} L(y) \right],$$

$$K_2(x, y) = \frac{l}{\sqrt{\lambda}xy} - \frac{1 + \sqrt{\lambda}}{(x + \sqrt{\lambda})(y + \sqrt{\lambda})} \frac{1}{\lambda^{1/4}} \ln \frac{1 + \lambda^{1/4}}{1 - \lambda^{1/4}} - \frac{1}{x - y} \left[\frac{L(x)}{x(x + \sqrt{\lambda})} - \frac{L(y)}{y(y + \sqrt{\lambda})} \right],$$

$$\varphi_1(x) = \frac{\text{Im} \bar{F}_1(x)}{(x - \alpha)},$$

$$\varphi_2(x) = \frac{\text{Im} \bar{F}_2(x)}{(x - \alpha)}; \quad (53)$$

the remaining notation is the same as in the boson case [Eq. (13)]. The condition (51) can be written in the form

$$H = \frac{1}{2} \frac{1}{1 + \sqrt{\lambda}} \pm \frac{1}{\pi} \int_1^{\infty} \frac{\varphi_1(x)}{x + \sqrt{\lambda}} dx - \frac{1}{\pi} \int_1^{\infty} \frac{\varphi_2(x)}{x + \sqrt{\lambda}} dx = 0. \quad (54)$$

When in the expression (52) we vary $\varphi_1(x)$ and $\varphi_2(x)$ subject to the condition (54) we get the following equations for the functions $\varphi_1(x)$ and $\varphi_2(x)$:

$$\frac{(x-1)^{3/2}(x-\lambda)^{3/2}}{x^2(x + \sqrt{\lambda})} \varphi_1(x) + \frac{\sqrt{\alpha}}{4\pi} [R_1(x) + (1 + \sqrt{\lambda})^2 R_2(x)] + \frac{1}{2\pi^2} \int_1^{\infty} [K_1(x, y) - (1 + \sqrt{\lambda})^2 K_2(x, y)] \times \varphi_1(y) dy \mp \frac{v}{4\pi} \frac{1}{x + \sqrt{\lambda}} = 0, \quad (55)$$

$$\frac{(x-1)^{1/2}(x-\lambda)^{1/2}}{x(x + \sqrt{\lambda})} \varphi_2(x) - \frac{1}{4\pi} \frac{(1 \pm \sqrt{\alpha})(\sqrt{\lambda} \pm \sqrt{\alpha})}{1 + \sqrt{\lambda}} R_2(x) + \frac{1}{2\pi^2} \int_1^{\infty} K_2(x, y) \varphi_2(y) dy + \frac{v}{4\pi} \frac{1}{x + \sqrt{\lambda}} = 0. \quad (56)$$

The undetermined Lagrangian multiplier ν is to be found from Eq. (54). The minimum value of Φ is expressed in terms of the solution of the equations (54)–(56) in the following way:

$$\Phi_{\min} = C + \frac{1}{2\pi} \int_1^\infty \{ \sqrt{\alpha} R_1(x) \varphi_1(x) + R_2(x) (1 + \sqrt{\lambda}) [\sqrt{\alpha} (1 + \sqrt{\lambda}) \varphi_1(x) - (1 \pm \sqrt{\alpha}) (\sqrt{\lambda} \pm \sqrt{\alpha}) \varphi_2(x)] \} dx. \quad (57)$$

The equations for the functions φ_1 and φ_2 have one important feature in which they differ from the corresponding equations in the boson case: for $x \rightarrow 1$ the coefficient of $\varphi_1(x)$ in Eq. (55) goes to zero as $(x-1)^{1/2}$. Because of this the equation (55) is singular and acquires meaning only if we suppose that the lower limit in the integral term in Eq. (55) is $1 + \epsilon$, where ϵ is a small quantity. It can be shown that for $\epsilon \rightarrow 0$ the quantity $\Phi_{\min}$ of Eq. (57) goes to a finite, nonzero limit, although the solution $\varphi_1(x)$ loses its meaning for $\epsilon = 0$.⁷⁾ This means that the minimum of Φ is realized through a sequence of functions, in which the limiting function, while satisfying all the requirements of analyticity, does not satisfy the dispersion relation. (That this is so can be seen below, when we calculate $\Phi_{\min}$ by an analytic method.) Thus the question remains open as to whether the value of $\Phi_{\min}$ which will be obtained corresponds to an actual physical situation, i.e., whether the minimum of Φ has not been set too low (and the maximum value of g^2 too high).

In the nonrelativistic (scalar) case ($1 - \alpha \ll 1$) the difficulties just pointed out are unimportant, and the problem can be solved easily. As follows from Eq. (57), for $1 - \alpha \ll 1$ the function $\varphi_2(x)$ is the important one, and for it we have

$$\varphi_2(x) = \frac{\sqrt{1-\alpha}}{2(x-\alpha)\sqrt{x-1}}. \quad (58)$$

For $\Phi_{\min}$ we get

$$\Phi_{\min} = \frac{\pi}{4} \sqrt{\frac{1-\lambda}{1-\alpha}} (1 + \sqrt{\lambda}), \quad (59)$$

which leads to the following restriction on g^2 :

$$g^2 < 4 \sqrt{\Delta/2\mu} m_a/m_b, \quad \mu = m_b m_c / (m_b + m_c). \quad (60)$$

The formula (60) is equivalent to the formula (22) for the boson case, and goes over into that formula when we recall the fact that the normalization of the fermion wave functions differs from that of the boson wave functions by a factor $(2m)^{1/2}$.

We shall now calculate $\Phi_{\min}$ by an analytical method.^[9-11] We make the conformal transfor-

mation (24). Then the path of integration goes over into the unit circle and Eq. (49) can be written in the form

$$\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \{ f_1(\theta) |\bar{F}_1(z)|^2 + f_2(\theta) |\bar{F}_2(z)|^2 \}, \quad z = e^{i\theta}, \quad (61)$$

where $[u = \tan^2(\theta/2)]$

$$f_1(\theta) = \pi \sqrt{1-\alpha} \frac{u^2}{1+u} \frac{[1-\lambda + (1-\alpha)u]^{1/2}}{[1 + (1-\alpha)u]^2 [1 + \sqrt{\lambda} + (1-\alpha)u]},$$

$$f_2(\theta) = \frac{\pi(1 + \sqrt{\lambda})^2}{\sqrt{1-\alpha}} \frac{u}{1+u} \frac{[1-\lambda + (1-\alpha)u]^{1/2}}{[1 + (1-\alpha)u] [1 + \sqrt{\lambda} + (1-\alpha)u]}. \quad (62)$$

The functions $\bar{F}_1(z)$ and $\bar{F}_2(z)$ can be expanded inside the unit circle $|z| < 1$ in terms of systems of polynomials $\psi_n^{(1)}(z)$ and $\psi_n^{(2)}(z)$ which are orthogonal on the unit circle with the respective weight functions $f_1(\theta)$ and $f_2(\theta)$:

$$\bar{F}_1(z) = \sum_n a_n \psi_n^{(1)}(z), \quad \bar{F}_2(z) = \sum_n b_n \psi_n^{(2)}(z). \quad (63)$$

[We note that the conditions for the expansions (63) to hold for $|z| = 1$ are weaker than the conditions for existence of the dispersion relations for the functions $\bar{F}_1$ and $\bar{F}_2$ (cf. [10].) When Eq. (63) is substituted in Eq. (61) the integral Φ takes the form

$$\Phi = \sum_n (a_n^2 + b_n^2), \quad (64)$$

and the conditions (51), (50) can be written

$$\mp \bar{F}_1(z_0) + \bar{F}_2(z_0) = 0, \quad z_0 = -\frac{(1 + \sqrt{\lambda})^{1/2} - (1 - \alpha)^{1/2}}{(1 + \sqrt{\lambda})^{1/2} + (1 - \alpha)^{1/2}}, \quad (65)$$

$$\bar{F}_1(0) - \frac{\sqrt{\alpha}}{2} = 0, \quad \bar{F}_2(0) - \frac{1}{2} \left[\frac{\alpha + \sqrt{\lambda}}{1 + \sqrt{\lambda}} \pm \sqrt{\alpha} \right] = 0. \quad (66)$$

When we look for the minimum of the expression (64) under the conditions (65), (66) we easily find

$$a_n = \frac{1}{2} [\nu_1 \psi_n^{(1)}(0) \mp \nu_3 \psi_n^{(1)}(z_0)],$$

$$b_n = \frac{1}{2} [\nu_2 \psi_n^{(2)}(0) + \nu_3 \psi_n^{(2)}(z_0)], \quad (67)$$

where ν_1, ν_2, ν_3 are Lagrangian undetermined multipliers. Substituting Eq. (67) in Eqs. (65), (66) and using the formula for calculating sums of orthogonal polynomials (see [10], page 310)

$$\sum_n \psi_n^*(z_1) \psi_n(z_2) = \frac{1}{1 - z_1^* z_2} \frac{1}{D^*(z_1) D(z_2)}, \quad (68)$$

$$D(z) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln f(\theta) \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\theta \right\}, \quad (69)$$

we get the following system of equations for the determination of ν_1, ν_2, ν_3 :

$$\nu_1 D_1^{-2}(0) \mp \nu_3 D_1^{-1}(0) D_1^{-1}(z_0) = \sqrt{\alpha},$$

$$\nu_2 D_2^{-2}(0) + \nu_3 D_2^{-1}(0) D_2^{-1}(z_0) = \frac{\alpha + \sqrt{\lambda}}{(1 + \sqrt{\lambda}) \pm \sqrt{\alpha}},$$

⁷⁾The authors are grateful to E. M. Landis for a discussion of this point.

$$\mp \nu_1 D_1^{-1}(0) D_1^{-1}(z_0) + \nu_2 D_2^{-1}(0) D_2^{-1}(z_0) + \nu_3 [D_1^{-2}(z_0) + D_2^{-2}(z_0)] / (1 - z_0^2) = 0. \quad (70)$$

Substituting Eq. (67) in Eq. (64) and using Eq. (68), we find

$$\begin{aligned} \Phi_{\min} = & \frac{1}{4} [\nu_1^2 D_1^{-2}(0) \mp 2\nu_1 \nu_3 D_1^{-1}(0) D_1^{-1}(z_0) + \nu_3^2 D_1^{-2}(z_0) / (1 - z_0^2)] \frac{1}{4} [\nu_2^2 D_2^{-2}(0) \\ & + 2\nu_2 \nu_3 D_2^{-1}(0) D_2^{-1}(z_0) + \nu_3^2 D_2^{-2}(z_0) / (1 - z_0^2)]. \end{aligned} \quad (71)$$

For the $f_1(\theta)$ and $f_2(\theta)$ found from Eq. (62) the functions $D_1(z)$ and $D_2(z)$ can be calculated easily, and are found to be $[v = (1 - z)/(1 + z)]$

$$\begin{aligned} D_1(z) = & \frac{1}{2} \sqrt{\frac{\pi}{1 - \alpha}} \frac{(1 - z)^2}{1 + z} \\ & \times \frac{[V(1 - \lambda)/(1 - \alpha) + v]^{1/2}}{[1/V(1 - \alpha + v)]^2 [(1 + V\bar{\lambda})/(1 - \alpha)]^{1/2} + v}, \\ D_2(z) = & \frac{V\pi(1 + V\bar{\lambda})}{1 - \alpha} \frac{1 - z}{2} \\ & \times \frac{[V(1 - \lambda)/(1 - \alpha) + v]^{1/2}}{[1/V(1 - \alpha + v)]^2 [(1 + V\bar{\lambda})/(1 - \alpha)]^{1/2} + v}. \end{aligned} \quad (72)$$

In order to determine $\Phi_{\min}$ we still have to solve the equations (70) for ν_1, ν_2, ν_3 , substitute these solutions in Eq. (71), and express the functions D by using Eq. (72). The final result obtained for $\Phi_{\min}$ is

$$\begin{aligned} \Phi_{\min} = & \frac{\pi}{16} \frac{(V\bar{1} - \alpha + V\bar{1} - \bar{\lambda})^3}{(1 + V\bar{1} - \alpha)^2 [V\bar{1} - \alpha + (1 + V\bar{\lambda})^{1/2}]^2} \\ & \times \left\{ \frac{\alpha V\bar{1} - \alpha}{(1 + V\bar{1} - \alpha)^2} + \frac{(1 \pm V\bar{\alpha})^2 (V\bar{\lambda} \pm V\bar{\alpha})^2}{V\bar{1} - \alpha (V\bar{1} - \alpha + V\bar{1} - \bar{\lambda})^2} \right. \\ & + 2 \frac{[V\bar{1} - \bar{\lambda} + (1 + V\bar{\lambda})^{1/2}]^2 [V\bar{1} - \alpha + (1 + V\bar{\lambda})^{1/2}]^2}{(\alpha + V\bar{\lambda})^2 [V\bar{1} - \bar{\lambda} + 2(1 + V\bar{\lambda})^{1/2} + 1 + V\bar{\lambda}]} \\ & \times \left[\frac{\alpha + V\bar{\lambda}}{V\bar{1} - \alpha + V\bar{1} - \bar{\lambda}} \pm \sqrt{\alpha} \left(\frac{1 + V\bar{\lambda}}{V\bar{1} - \alpha + V\bar{1} - \bar{\lambda}} \right. \right. \\ & \left. \left. - \frac{V\bar{1} - \alpha (1 + V\bar{\lambda})^{1/2} [1 + (1 + V\bar{\lambda})^{1/2}]}{[V\bar{1} - \bar{\lambda} + (1 + V\bar{\lambda})^{1/2}] (1 + V\bar{1} - \alpha)} \right) \right]^2 \left. \right\}. \end{aligned} \quad (73)$$

In the scalar nonrelativistic case ($1 - \alpha \ll 1$) the expression (73) goes over into (59). When we substitute in Eq. (73) the numerical values of the masses for the most interesting case—the interaction of π mesons with nucleons (particles a and b are nucleons and c is a π meson)—we get $\Phi_{\min} = 0.0245$. From this we get as the restriction on the pion-nucleon coupling constant (the coefficient $1/3$ is used on account of the isotopically symmetrical theory)

$$g^2 < 2\pi/3 \Phi_{\min} = 85 \quad (74)$$

or

$$f^2 = (\mu/2m)^2 g^2 < 0.47. \quad (75)$$

For the $\Sigma\Lambda\pi$ interaction (a is the fermion Σ ,

b is the fermion Λ , and c is the pion), if the parities of Σ and Λ are different we have $\Phi_{\min} = 1.95$ and

$$g_{\Sigma\Lambda\pi}^2 < 3.2. \quad (76)$$

For like parities of the Σ and Λ hyperons $\Phi_{\min} = 0.0120$ and $g_{\Sigma\Lambda\pi}^2 < 520$. Finally, for the $\Xi\Xi\pi$ interaction $\Phi_{\min} = 0.0160$ and $g_{\Xi\Xi\pi}^2 < 130$.

It is interesting to see how the restriction on the constant for the $\Sigma\Lambda\pi$ interaction is altered if we take Λ instead of Σ as fermion a . Calculating $\Phi_{\min}$ we find that in this case $g_{\Sigma\Lambda\pi}^2 < 7.5$ if Σ and Λ have opposite parities, and $g_{\Sigma\Lambda\pi}^2 < 260$ if the parities are the same, i.e., as the binding energy increases the maximum value of g^2 increases for the scalar interaction and decreases for the pseudoscalar interaction.⁸⁾

CONCLUSION

We have found that for prescribed values of the masses the coupling constants in quantum field theory are bounded.⁹⁾ The fact that the value of the coupling constant g^2 cannot exceed a certain value $g_{\max}^2$ means that the amplitudes of the quantum field theory, considered as functions of g^2 , have a singular point at $g^2 = g_{\max}^2$, and for $g^2 > g_{\max}^2$ the theory will be physically contradictory.

It is interesting to discuss what may happen if g^2 is larger than $g_{\max}^2$. One possibility is that the vertex part $\Gamma(\kappa^2)$ will have a pole $\kappa^2 = m_0^2$ which lies in the interval $m_a^2 < m_0^2 < (m_b + m_c)^2$, i.e., that a new physical state with the mass m_0 will appear.

Another possibility is that $D^{-1}(\kappa^2)$ will cease to be an R -function, i.e., for example one of the constants R or α in Eq. (2) will be negative. [When there is only one pole of $D(\kappa^2)$ only one of the constants R_n in Eq. (2) can be different from zero.] It is not hard to verify that in addition to the zero of $D^{-1}(\kappa^2)$ at $\kappa^2 = m_a^2$ there will be another zero at $\kappa^2 = \kappa_1^2$, with $\kappa_1^2 < m_a^2$ and $[D^{-1}(\kappa^2)]'_{\kappa^2=\kappa_1^2} = 0$. Consequently, in this case the Green's function $D(\kappa^2)$ has a pole at $\kappa^2 = \kappa_1^2$ with

⁸⁾All of the numerical values of $\Phi_{\min}$ given in the text have also been obtained by numerical solution of the equations (54)–(56) (by cutting the integrals off at the lower limit and afterward letting the cut-off limit approach zero) and calculation of $\Phi_{\min}$ from Eq. (57).

⁹⁾In the problem of the interaction of neutral scalar mesons with a static nucleon, which, as is well known, has an exact solution and does not lead to difficulties of the zero-charge type, there has been an earlier (and rigorous) proof of the fact that the coupling constant is bounded (see [12]).

a negative residue, i.e., an unphysical state with negative norm (a "ghost") will appear.

The fact that the coupling constants in quantum field theory have upper bounds indicates that all attempts to approach the solution of problems by using expansions in powers of $1/g^2$ with fixed masses (strong-coupling theory) have extremely little chance of success.

We shall also make a comment on a paper by Landau.^[13] In this paper Landau put forward the idea that the constant g^2 must be equal to the value determined for it for the case in which particle a is represented as a bound state of particles b and c with a point interaction. It follows from our treatment that in general this assertion cannot be correct, since all the other states besides the two-particle states make positive contributions to the relation (5).

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ANALYTIC EXPRESSIONS FOR UPPER LIMITS OF COUPLING CONSTANTS IN QUANTUM FIELD THEORY

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Analytic expressions are obtained for upper limits of the coupling constant for three fields a , b , and c as a function of the particle masses m_a , m_b , and m_c .

1. The coupling constant g^2 of three fields a , b , c satisfies, as shown by Lehmann, Symanzik, and Zimmermann,^[1] the inequality $g^2\Phi(a) < 1$, where $\Phi(a)$ is a certain functional of particle a . From here follows the inequality $g^2 < 1/\Phi_{\min}$, which however is useful only if the condition $\Phi_{\min} \neq 0$ is satisfied. Geshkenbein and Ioffe^[2] obtained an analogous inequality with a functional satisfying this condition. The form of this functional depends on the properties of particle a and on the type of the reaction describing the transition of particle a into particles b and c . For example for a boson $\Phi(a)$ has one form and for a fermion it has a substantially different form. If a is a fermion then the functional Φ differs somewhat for the scalar and pseudoscalar cases.^[2]

In a conversation with me B. L. Ioffe has posed the question of the possibility of an analytical solution of the extremum problem for the functional Φ . In the present paper the solution of this problem is given and, consequently, analytic expressions are obtained for upper limits of coupling constants.

Let particle a be a boson of mass m_a . As was shown in^[2] the coupling constant of the three fields a , b , and c satisfies the inequality

$$g^2 \int_{(m_b+m_c)^2}^{\infty} \frac{1}{2\pi} \frac{|\Gamma(\kappa^2)|^2 q(\kappa^2) d\kappa^2}{(\kappa^2 - m_a^2)^2 \kappa} < 1, \quad (1.1)$$

where the masses m_b and m_c are such that for all possible transitions of particle a into particles b and c the sum $(m_b + m_c)$ is closest to m_a , $m_a < m_b + m_c$. $\Gamma(\kappa^2)$ is the vertex part for the transition of the boson a into the bosons b and c , and $q(\kappa^2) = [\kappa^2 - (m_b + m_c)^2]^{1/2} [\kappa^2 - (m_b - m_c)^2]^{1/2} / 2\kappa$.

In terms of the dimensionless variable $x = \kappa^2 / (m_b + m_c)^2$ the inequality (1.1) becomes

$$g^2 \frac{1}{2(m_b + m_c)^2} \int_1^{\infty} \frac{1}{2\pi} \frac{|\Gamma(x)|^2 \sqrt{(x-1)(x-\lambda)}}{(x-\alpha)^2 x} dx < 1, \quad (1.2)$$

where

$$\alpha = m_a^2 / (m_b + m_c)^2, \quad \lambda = (m_b - m_c)^2 / (m_b + m_c)^2, \quad \alpha < 1, \quad \alpha < \lambda. \quad (1.3)$$

The inequality (1.1) is a consequence of the inequality

$$\int_{(m_b+m_c)^2}^{\infty} \text{Im } D^{-1}(\kappa^2) \frac{d\kappa^2}{(\kappa^2 - m_a^2)^2} \leq 1, \quad (1.4)$$

where $D(\kappa^2)$ is the Green's function of the boson a . We shall derive this inequality because the derivation presented in the paper of Geshkenbein and Ioffe^[2] does not seem to us to be sufficiently convincing.

From the Lehmann-Källén representation^[3]

$$D(\kappa^2) = \frac{1}{\kappa^2 - m_a^2} - \int_{(m_b+m_c)^2}^{\infty} \frac{\rho(\kappa'^2)}{\kappa'^2 - \kappa^2 - i\delta} d\kappa'^2, \quad \rho(\kappa'^2) \geq 0 \quad (1.5)$$

it follows that $D(\kappa^2)$, hence also $D^{-1}(\kappa^2)$, is an R-function.

An arbitrary R-function $F(x)$ has the representation^[4]

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + xx'}{x' - x} d\sigma(x') + \text{Re } F(i), \quad (1.6)$$

where $d\sigma(x') \geq 0$ is some distribution of nonnegative masses with a finite total mass (see Appendix I)

$$\int_{-\infty}^{\infty} d\sigma(x') = 2\pi \text{Im } F(i) < +\infty. \quad (1.7)$$

Let us separate out from Eq. (1.6) all the delta-function terms. Then

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + xx'}{x' - x} d\sigma_1(x') + \text{Re } F(i) + \frac{\mu_{\infty}}{2\pi} x + \frac{1}{2\pi} \sum_k \frac{1 + xx_k}{x_k - x} \mu_k, \quad (1.8)$$

where $\mu_k > 0$ is the coefficient of $\delta(x' - x_k)$, ($\mu_\infty = 0$ if there is no delta-function term at ∞), $d\sigma_1(x') \geq 0$ and

$$\mu_\infty + \sum \mu_k + \int_{-\infty}^{\infty} d\sigma_1(x') = 2\pi \operatorname{Im} F(i). \quad (1.9)$$

It follows from Eq. (1.6) that for any interval of the real axis for an arbitrary function $f(x)$ one has

$$\lim_{\delta \rightarrow +0} \int \frac{f(x) 2 \operatorname{Im} F(x' + i\delta) dx'}{1 + x'^2} = \int f(x') d\sigma(x'). \quad (1.10)$$

In particular,

$$F'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1+x'^2)}{(x'-x)^2} d\sigma(x') = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} F(x' - i\delta)}{(x'-x)^2} dx'. \quad (1.11)$$

Since $D^{-1}(\kappa^2)$ is an R-function which is real for $\kappa^2 \leq (m_b + m_c)^2$ and whose derivative at the point m_a^2 equals 1, it follows from Eq. (1.11) that

$$\int_{(m_b+m_c)^2}^{\infty} \operatorname{Im} D^{-1}(\kappa^2) \frac{d\kappa^2}{(\kappa^2 - m_a^2)} \leq 1,$$

i.e., the inequality (1.4). The equality became an inequality because the integral extends not from $-\infty$ to $+\infty$ but from $(m_b + m_c)^2$ to $+\infty$. The fact that $\operatorname{Im} D^{-1}(\kappa^2) = 0$ on the interval $(-\infty, (m_b + m_c)^2)$ does not change matters since the function $D^{-1}(\kappa^2)$ may have poles in that interval. It is easy to convince oneself that each pole gives a contribution $\mu_k(1 + x_k^2)/(x_k - x)^2$ to the integral (1.11).

From the representation (1.6) it is easily established that the R-function satisfies a dispersion relation with one subtraction (see Appendix I).

2. Let us denote the integral in Eq. (1.2) by $\Phi(\Gamma)$. The problem consists in finding the minimum of this functional; at that it is necessary to define precisely the class of functions on which the minimum is being sought. In correspondence with the work of Geshkenbein and Ioffe, it is supposed that $\Gamma(x)$ is a function that is regular in the plane with the cut $(1, +\infty)$ and is real for $x < 1$, consequently $\Gamma(x)$ takes on complex conjugate values on opposite sides of the cut. We shall assume that at ∞ the function $\Gamma(x)$ grows slower than any power of $|x|^{1/2}$, i.e., that for any $\epsilon > 0$ for sufficiently large $|x|$ the function $|\Gamma(x)| < \exp(\epsilon |x|^{1/2})$. The function $\Gamma(x)$ is normalized by the condition $\Gamma(\alpha) = 1$.

Let us map conformally the cut x -plane onto the unit circle $|z| \leq 1$. We let

$$z = -\frac{t-i}{t+i}, \quad t = \sqrt{\frac{x-1}{1-\alpha}}. \quad (2.1)$$

This mapping takes the lower edge of the cut into

the lower half circumference, the upper edge into the upper half circumference, the point $x = \alpha$ into the center $z = 0$, the point $x = \infty$ into the point $z = -1$, and the functional $\Phi(\Gamma)$ takes on the form

$$\Phi(\Gamma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) |\Gamma[x(z)]|^2 d\theta, \quad z = e^{i\theta} \quad (2.2)$$

with the weight function

$$f(\theta) = \frac{1}{2} \frac{1}{\sqrt{1-\alpha}} \frac{t^2 \sqrt{(1-\alpha)t^2 + (1-\lambda)}}{[(1-\alpha)t^2 + 1](t^2 + 1)}, \quad t = \tan \frac{\theta}{2}. \quad (2.3)$$

The family of functions inside the circle $|z| < 1$, on which the minimum of the functional $\Phi(\Gamma)$ is being sought, is characterized by the following properties: 1) the $\Gamma(z)$ are regular in the closed circle $|z| \leq 1$ with the point $z = -1$ excluded, $\Gamma(0) = 1$ and $\Gamma(z^*) = (\Gamma(z))^*$; 2) for any $\epsilon > 0$ in a sufficiently small neighborhood of the point $z = -1$, $|\Gamma(z)| < \exp(\epsilon/|1+z|)$.

Szego^[5] and Smirnov^[6] have completely solved the general extremum problem for the functional $\int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta$, where $p(\theta)$ is a prescribed nonnegative function and $F(z)$ is an arbitrary function from the class H_2 (about the class H_2 see below). We will show that the result of Szego and Smirnov applies to our case as well, but will obtain the solution by a different method in order to, in contrast to the solution of Szego and Smirnov, make it directly applicable to the case of a functional of several functions, subject to several linear relations. It is precisely this kind of extremum problem that must be solved if the particle a is a fermion.

3. Let $p(\theta)$ be a nonnegative function and the integral $\int_{-\pi}^{\pi} p(\theta) d\theta < +\infty$. Under these conditions there exists a set of orthonormal polynomials $\varphi_n(z)$ with the weight $p(\theta)$, i.e., a set of polynomials satisfying the conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) \varphi_n(e^{i\theta}) \varphi_m^*(e^{-i\theta}) d\theta = \delta_{nm}. \quad (3.1)$$

This set of polynomials is unique if one imposes the additional requirement that the coefficient of the highest power z^n in $\varphi_n(z)$ be positive. If the

⁵⁾We note that the minimum of the integral (2.2) on the class of all regular functions (z) , normalized by the condition $\Gamma(0) = 1$, is equal to zero. This can be seen by setting

$$\Gamma_n(z) = \exp \left[n \frac{1-z}{1+z} - n \right], \quad |\Gamma_n(e^{i\theta})| = e^{-n}$$

then as $n \rightarrow +\infty$ the function $\Phi(\Gamma_n) \rightarrow 0$.

weight function $p(\theta)$ is an even function then all the coefficients in the polynomials $\varphi_n(z)$ are real.^[5] We shall consider only such weight func-

tions $p(\theta)$ for which the integrals $\int_{-\pi}^{\pi} |\ln p(\theta)| d\theta$ exist, so that the harmonic function which is equal to $\ln p(\theta)$ on the circumference is given by the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln p(\theta) \left\{ \frac{1-r^2}{1+r^2-2r \cos(\theta-\varphi)} \right\} d\theta, \quad z = re^{i\varphi}.$$

The Poisson kernel $\{ \dots \}$ is equal to the real part of the Schwarz kernel $(e^{i\theta} + z)/(e^{i\theta} - z)$, consequently the integral $\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln p(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta$ is a regular function inside the circle $|z| < 1$, whose real part tends to $\ln p(\theta)$ as $z \rightarrow e^{i\theta}$. It then follows that the regular inside the unit circle function

$$D(z) = \exp \left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln p(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right) \quad (3.2)$$

possesses the following properties: 1) $D(z) \neq 0$ for $|z| < 1$; 2) $|D(re^{i\theta})|^2 \rightarrow p(\theta)$ as $r \rightarrow 1$; 3) $D(0) > 0$. The function $D(z)$ plays a fundamental role in the solution of the extremum problem. In Appendix II the following identity will be proved:

$$\sum_{n=0}^{\infty} \varphi_n^*(a^*) \varphi_n(z) = \frac{1}{1-a^*z} \frac{1}{D^*(a^*)} \frac{1}{D(z)}. \quad (3.3)$$

In particular

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^2 = \frac{1}{1-|z|^2} \frac{1}{|D(z)|^2}, \quad \sum_{n=0}^{\infty} |\varphi_n(0)|^2 = D^{-2}(0). \quad (3.4)$$

A function $\psi(z)$ regular inside the unit circle $= \sum_{n=0}^{\infty} c_n z^n$ is said to be a function of the class H_2 if $\sum |c_n|^2 < +\infty$. It is in terms of this class of functions that one solves the problem of the conditions necessary in order that the Fourier series of a function $F(z)$ converge to that function. In Appendix II will be proved the property established by Smirnov^[6]: the Fourier series in the polynomials $\varphi_n(z)$ of a function $F_n(z)$ regular inside the unit circle converges uniformly to the function $F(z)$ in an arbitrary inner circle if and only if the product $D(z)F(z)$ belongs to the class H_2 . At that the condition of completeness is satisfied

$$\sum |F_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta, \quad F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) F(e^{i\theta}) \varphi_n^*(e^{-i\theta}) d\theta. \quad (3.5)$$

From here one immediately obtains the solution to the problem of finding the minimum of the functional

$$\Phi(F; p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta \quad (3.6)$$

subject to the relation $F(0) = C$, on the family of functions such that the product $D(z)F(z)$ belongs to the class H_2 . Indeed, it follows from Eq. (3.5) that one must find the minimum of the sum of the series

$$\sum_{n=0}^{\infty} |F_n|^2 = \sum_{n=0}^{\infty} F_n F_n^* \quad (3.7)$$

subject to the linear relation

$$\sum_{n=0}^{\infty} \varphi_n(a) F_n = C = C_1 + iC_2. \quad (3.8)$$

In the general case, when the quantities appearing in Eq. (3.8) are complex, one obtains two linear relations

$$\sum (\varphi_n F_n + \varphi_n^* F_n^*) = 2C_1, \quad -i \sum (\varphi_n F_n - \varphi_n^* F_n^*) = 2C_2. \quad (3.8')$$

Introducing the Lagrange multipliers λ_1 and λ_2 we find that for the extremum function

$$F_n = (\lambda_1 + i\lambda_2) \varphi_n^*(a^*). \quad (3.9)$$

On substitution in Eq. (3.8) we obtain

$$\lambda_1 + i\lambda_2 = (C_1 + iC_2) / \sum_{n=0}^{\infty} |\varphi_n(a)|^2. \quad (3.10)$$

It follows from Eqs. (3.4) and (3.3) that

$$\Phi_{\min} = (\lambda_1^2 + \lambda_2^2) \sum |\varphi_n(a)|^2 = |C|^2 (1 - |a|^2) |D(a)|^2, \quad \Gamma_{\text{extr}}(z) = (\lambda_1 + i\lambda_2) \sum \varphi_n(a) \varphi_n(z) = C \frac{1 - |a|^2 D(a)}{1 - a^* z D(z)}. \quad (3.11)$$

If the quantities F_n , a , $\varphi_n(a)$ and C are real then the relation (3.8) defines a hyperplane in the Hilbert space of the coefficients $\{F_n\}$, and $\Phi_{\min}$ is nothing else but the square of the distance from the origin of the coordinate system to that hyperplane.

4. For the extremum problem of Sec. 2 the weight function $p(\theta)$ is equal to $f(\theta)$, Eq. (2.3), $a = 0$, $C = 1$. For this weight function²⁾

$$D(0) = \left(1 + \sqrt{\frac{1-\lambda}{1-\alpha}} \right)^{1/2} / 2 \sqrt{2} (1 + \sqrt{1-\alpha}), \quad (4.1)$$

$$D(z) = \frac{(1-z)(1+z)^{1/2}}{2 \sqrt{2} (1-\alpha)^{1/4}} \times \frac{[\sqrt{1-\lambda} + \sqrt{1-\alpha} + z(\sqrt{1-\lambda} - \sqrt{1-\alpha})]^{1/2}}{1 + \sqrt{1-\alpha} + z(1 - \sqrt{1-\alpha})}. \quad (4.2)$$

²⁾The integral giving $D(0)$ can be found in any table of integrals, the integral for $D(z)$ is evaluated by Geshkenbein and Ioffe.

If for any $\epsilon > 0$ within a neighborhood sufficiently close to the point $z = -1$ the function $\Gamma(z)$ is less than $\exp(\epsilon/|1+z|)$ then the same is true of the product $D(z)\Gamma(z)$. In addition

$$\int_{-\pi}^{\pi} |D(e^{i\theta})|^2 |\Gamma(e^{i\theta})|^2 d\theta = \int_{-\pi}^{\pi} f(\theta) |\Gamma(e^{i\theta})|^2 d\theta < +\infty.$$

In Appendix III it will be shown that it follows from these two properties that the product $D(z)\Gamma(z)$ belongs to the class H_2 and therefore the general solution (3.11) is valid in this particular case as well. Since $a = 0$, $C = 1$ we have

$$\Phi_{min} = D^2(0), \quad \Gamma_{\text{KCT}} = D(0)/D(z). \quad (4.3)$$

If one substitutes into these expressions the values $D(0)$ and $D(z)$ from Eqs. (4.1) and (4.2) one obtains the following expressions for all extremal quantities:

$$\Gamma_{\text{extr}}(x) = \frac{1}{2} \frac{(\sqrt{1-\alpha} + \sqrt{1-\lambda})^{1/2}}{1 + \sqrt{1-\alpha}} \frac{1}{\sqrt{1-x}} \times \frac{(1+i\sqrt{x-1})(\sqrt{1-\alpha} - \sqrt{1-x})}{(\sqrt{1-\lambda} - \sqrt{1-x})^{1/2}}, \quad (4.4)$$

$$\Phi_{min} = \frac{1}{8\sqrt{1-\alpha}} \frac{\sqrt{1-\alpha} - \sqrt{1-\lambda}}{(1 + \sqrt{1-\alpha})^2}, \quad (4.5)$$

$$g^2 < 16 \frac{[(m_b + m_c)^2 - m_a^2]^{1/2} [(m_b + m_c) + \sqrt{(m_b + m_c)^2 - m_a^2}]^2}{2\sqrt{m_b m_c} + \sqrt{(m_b + m_c)^2 - m_a^2}}. \quad (4.6)$$

The care that must be exercised when taking the limit in solving extremum problems is illustrated by the following circumstance, interesting in its own right.

Let us suppose that the weight function tends to zero on some interval. Then $D(0)$ and Φ_{min} also tend to zero. It follows that if the lower limit in the integral defining the functional Φ should lie some distance to the right of the beginning of the cut then $\Phi_{min} = 0$, no matter how small that distance might be.

5. For physical applications a somewhat more general extremum problem is of interest, namely to find the minimum of the functional $\Phi(F_1, F_2, \dots, F_k)$ which depends on k functions in the form

$$\Phi = \sum_{\nu=1}^k \frac{1}{2\pi} \int_{-\pi}^{\pi} p_{\nu}(\theta) |F_{\nu}(e^{i\theta})|^2 d\theta \quad (5.1)$$

subject to several linear relations

$$\sum_{\nu=1}^k a_{i\nu} F_{\nu}(z_i) = b_i, \quad i = 1, 2, \dots, l, \quad |z_i| < 1. \quad (5.2)$$

It is assumed that all $p_{\nu}(\theta)$ are nonnegative functions that satisfy the conditions formulated in

Sec. 3. The minimum of the functional is sought on a family of functions regular inside the unit circle, with all products $D_{\nu}(z)F_{\nu}(z)$ belonging to the class H_2 . By $D_{\nu}(z)$ we mean the function $D(z)$ corresponding to the weight function $p_{\nu}(z)$.

This problem is solved in precisely the same way as the extremum problem for one function with one linear relation. Each of the functions $F_{\nu}(z)$ is expanded in the Fourier series $\sum F_{\nu n} \varphi_n^{(\nu)}(z)$ in the orthogonal polynomials corresponding to the weight function $p_{\nu}(\theta)$. The functional Φ goes over into the sum

$$\Phi = \sum_{n=0}^{\infty} (|F_{1n}|^2 + \dots + |F_{kn}|^2). \quad (5.3)$$

To each of the relations (5.2) there correspond two Lagrange multipliers λ_{i1} and λ_{i2} . If one introduces $\lambda_i = \lambda_{i1} + i\lambda_{i2}$ then for the extremum functions one has

$$F_{\nu n} = \sum_{i=1}^l \lambda_i a_{i\nu}^* \varphi_n^{(\nu)*}(z_i^*). \quad (5.4)$$

The multipliers λ_i are determined from the set of equations

$$\sum_{i=1}^l c_{ij} \lambda_j = b_i, \quad i = 1, 2, \dots, l, \quad (5.5)$$

where

$$c_{ij} = \frac{1}{1 - z_i^* z_j^*} \sum_{\nu=1}^k \frac{a_{i\nu}^* a_{j\nu}}{D_{\nu}(z_i) D_{\nu}^*(z_j^*)}, \quad c_{ii} = c_{ii}^*. \quad (5.6)$$

The solution of the extremum problem is expressed in terms of the c_{ij} as follows:

$$\Phi_{min} = \sum_{i=1}^l b_i^* \lambda_i = - \left\| \begin{array}{c} 0 \quad b_1^* \dots b_l^* \\ b_1 \quad c_{11} \dots c_{1l} \\ \dots \dots \dots \\ b_l \quad c_{l1} \dots c_{ll} \end{array} \right\| \|c_{ij}\|^{-1}, \quad (5.7)$$

and the extremal functions as follows

$$F_{\nu}(z) = \frac{1}{D_{\nu}(z)} \sum_{i=1}^l \frac{a_{i\nu}^* \lambda_i}{1 - z_i^* z} \frac{1}{D_{\nu}^*(z_i^*)}. \quad (5.8)$$

In the fermion problem $k = 2$, $l = 3$, and the weight functions are such that $D_1(z)$ and $D_2(z)$ can be evaluated explicitly. As a result a closed form expression can be given for Φ_{min} in terms of the particle masses m_a , m_b , and m_c . [2]

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APPENDIX I

An arbitrary positive function $v(\xi)$ harmonic inside the circle $|\xi| < 1$ can be represented in the form

$$v(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} d\psi(\theta), \quad \zeta = re^{i\varphi}, \quad (\text{I.1})$$

where $d\psi(\theta)$ is a certain distribution of nonnegative masses. We omit the proof as there can be no doubt as to the truth of this assertion. It is clear that the total mass

$$\int_{-\pi}^{\pi} d\psi(\theta) = 2\pi v(0) < +\infty. \quad (\text{I.2})$$

It follows from Eq. (I.1) that the analytic function $f(\zeta) = v(\zeta) - iu(\zeta)$, for which $v(\zeta)$ serves as its real part, can be represented in the form

$$f(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\psi(\theta) + i \operatorname{Im} f(0). \quad (\text{I.3})$$

In other words, a function regular inside the unit circle and with a positive real part can be expressed in the form Eq. (I.3).

This fact has at one time (1910) been established by Caratheodory and Herglotz. If one goes over from the unit circle $|\zeta| < 1$ to the upper half-plane $\operatorname{Im} x > 0$ by means of the transformation $\zeta + 1 = 2i/(x+1)$, and from the function $f(\zeta)$ to the R-function $F(x) = if(\zeta)$, then one obtains the representation, Eq. (1.6). From Eq. (1.6) and the identity

$$\frac{1+x'x}{x'-x} = \frac{1+x'^2}{x'-x} - x'$$

it follows that

$$F(x) = F(x_0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{x'-x} - \frac{1}{x'-x_0} \right] (1+x'^2) d\sigma(x'), \quad (\text{I.4})$$

but $(1+x'^2)d\sigma(x')$ may be replaced by $2 \operatorname{Im} F(x' + i\delta)dx'$ and therefore

$$F(x) = F(x_0) + \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{x'-x} - \frac{1}{x'-x_0} \right] \operatorname{Im} F(x' + i\delta) dx', \quad (\text{I.5})$$

i.e., every R-function satisfies a dispersion relation with no more than one subtraction. Let us note that in the relation (I.5) it is not assumed that the function $F(x)$ has no real poles. One has the relation

$$\int_{-\infty}^{\infty} \frac{2 \operatorname{Im} F(x' + i\delta)}{1+x'^2} dx' = \int_{-\pi}^{\pi} d\sigma(x') = 2\pi \operatorname{Im} F(i). \quad (\text{I.6})$$

Sometimes the following property of an R-function is useful: it satisfies the double inequality

$$\frac{\sin \varphi}{C} \frac{1}{r} |F(x)| < \frac{Cr}{\sin \varphi}, \quad x = re^{i\varphi}, \quad r > 1, \quad (\text{I.7})$$

where C is a constant that depends on the function (see [7], p. 72).

APPENDIX II

Functions of the class H_2 are characterized by the following property: if $f(z) = \sum_{\nu=0}^{\infty} c_{\nu} z^{\nu}$ then

$$\lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta = \sum_{\nu=0}^{\infty} |c_{\nu}|^2 < +\infty.$$

If $f(z)$ and $g(z)$ are functions of the class H_2 then

$$\lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{i\theta}) (g(\rho e^{i\theta}))^* d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) (g(e^{i\theta}))^* d\theta. \quad (\text{II.1})$$

It is easy to see that the function $D(z)$ belongs to the class H_2 (see [5], p. 285). Let us prove that if $\Sigma F_{\nu} \varphi_{\nu}(e^{i\theta})$ is the Fourier series of the function $F(e^{i\theta})$ in the polynomials $\varphi_{\nu}(t)$ and the product $D(z)F(z)$ belongs to the class H_2 then the completeness relation

$$\sum_{\nu=0}^{\infty} |F_{\nu}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\theta) |F(e^{i\theta})|^2 d\theta < +\infty \quad (\text{II.2})$$

holds. From completeness will follow the uniform convergence of $\Sigma F_{\nu} \varphi_{\nu}(z)$ to $F(z)$ inside any inner circle.

Let us denote by $\varphi_{\rho, \nu}(z)$ a set of orthonormal polynomials with the weight $|D(\rho e^{i\theta})|$. Since as $\rho \rightarrow 1$ the quantity $|D(\rho e^{i\theta})|^2$ tends to $\rho(\theta)$ it follows that for any fixed ν the function $\varphi_{\rho, \nu}(z)$ tends uniformly to $\varphi_{\nu}(t)$ in the closed circle $|z| \leq 1$. Let us denote by $F_{\rho, \nu}$ the coefficients in the Fourier expansion of the function $F(\rho e^{i\theta})$ with respect to this set of polynomials:

$$F_{\rho, \nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D(\rho e^{i\theta})|^2 F(\rho e^{i\theta}) \varphi_{\rho, \nu}^* d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} (D(\rho e^{i\theta}) F(\rho e^{i\theta})) (D(\rho e^{i\theta}) \varphi_{\nu}(e^{i\theta}))^* d\theta.$$

Since the products $D(\rho e^{i\theta}) F(\rho e^{i\theta})$ and $D(\rho e^{i\theta}) \varphi_{\nu}(\rho e^{i\theta})$ belong to the class H_2 we have, as a consequence of Eq. (II.1),

$$\lim_{\rho \rightarrow 1} F_{\rho, \nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (D(e^{i\theta}) F(e^{i\theta})) (D(e^{i\theta}) \varphi_{\nu}(e^{i\theta}))^* d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\theta) F(e^{i\theta}) \varphi_{\nu}^*(e^{-i\theta}) d\theta = F_{\nu}.$$

Thus the completeness relation is valid for the function $F(\rho e^{i\theta})$

$$\sum_{\nu=0}^{\infty} |F_{\rho, \nu}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D(\rho e^{i\theta})|^2 |F(\rho e^{i\theta})|^2 d\theta.$$

As $\rho \rightarrow 1$ the integral on the right tends to

$(2\pi)^{-1} \cdot \int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta$ since $D(z)F(z) \in H_2$ and, consequently,

$$\sum_{v=0}^{\infty} |F_v|^2 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta.$$

By the Bessel inequality

$$\sum_{v=0}^{\infty} |F_v|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta) |F(e^{i\theta})|^2 d\theta,$$

i.e., the equality sign holds.

Let us take for $F(z)$ the function $[(1 - a^*z) \times D(a)D(z)]^{-1}$, $|a| < 1$. It is easy to verify by direct calculation that $F_v = \varphi_v^*(a^*)$.

Since $D(z)F(z) = (1 - a^*z)^{-1}D(a)$ belongs to the class H_2 it follows that for $|z| < 1$

$$\sum_{v=0}^{\infty} \varphi_v^*(a^*) \varphi_v(t) = [(1 - a^*z) D(a) D(z)]^{-1}.$$

This identity is given by Szego (see [5], p. 311) and by Smirnov. [6] The difference lies in the fact that we solve the extremum problem on the basis of this identity whereas Szego and Smirnov prove the identity on the basis of the solution of the extremum problem.

APPENDIX III

Let $\psi(z) = \sum c_n z^n$ be a function regular in the closed circle $|z| \leq 1$ with the exception of the point $z = -1$ and satisfying the following conditions: 1) for any $\epsilon > 0$ in a sufficiently small neighborhood of the point $z = -1$ one has $|\psi(z)| < \exp(\epsilon/|1+z|)$; 2) $\int_{-\pi}^{\pi} |\psi(e^{i\theta})|^2 d\theta < +\infty$. Let us show that the function $\psi(z)$ belongs to the class H_2 , i.e., that $\sum |c_n|^2 < +\infty$.

The transformation $w + i = 2i/(1+z)$ maps the circle $|z| < 1$ into the upper half-plane $\text{Im } w > 0$; at that the point $z = -1$ goes to ∞ . We introduce the function $F(w) = i\sqrt{2} \psi(z)/(w+i)$. The function $F(w)$ is regular in the upper half-plane, for any $\epsilon > 0$ and sufficiently large $|w|$ the inequality $|F(w)| < \exp(\epsilon|w|)$ is satisfied and, finally, the integral $\int_{-\infty}^{\infty} |F(w)|^2 dw < +\infty$.

As was shown by Wiener and Paley (see [8], p. 9), for a function with these properties the integral over the large upper semicircle in the Cauchy formula tends to zero, i.e.,

$$F(w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(w')}{w' - w} dw'.$$

From here there follows for the function $\psi(z)$ the representation

$$\psi(z) = \frac{1}{2\pi i} \int_{|z'|=1} \frac{\psi(z') dz'}{z' - z}, \quad (\text{III.1})$$

I.e., the Cauchy formula is valid for the function $\psi(z)$ not only when the integration is over an internal circle but also when the integration is over the boundary circle.

It follows from Eq. (III.1) that

$$c_n = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \psi(e^{i\theta}) e^{-in\theta} d\theta,$$

i.e., c_n coincides with the Fourier coefficient of the function $\psi(e^{i\theta})$. In accordance with the Bessel inequality

$$\sum |c_n|^2 < \frac{1}{2\pi} \int_{-\pi}^{\pi} |\psi(e^{i\theta})|^2 d\theta < +\infty.$$

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SPIN-INVERSION LEVELS IN A MAGNETIC FIELD AND THE EPR SPECTRUM OF OCTAHEDRAL Cu^{2+} ION COMPLEXES

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With the octahedral Cu^{2+} ion complexes taken as an example, it is shown that by taking into account the inversion splitting previously investigated by the author, a system of spin-inversion (spin-electronic-vibrational) levels appears in a magnetic field. One of the features of the system is that some of the levels correspond to the same spin states. The transition probabilities between these states under the action of an electromagnetic perturbation depend significantly on the relation between the magnetic field strength and the magnitude of the inversion splitting. In this way one obtains a characteristic temperature transition from one type of EPR spectrum to another, which is confirmed by the available experimental data. An unusual frequency dependence of the spectrum and, in particular, the possibility of absorption in zero field is also deduced.

1. INTRODUCTION

IN a previous paper^[1] we considered the inversion splitting of the electronic-vibrational levels of an octahedral complex arising from the presence of several equivalent equilibrium configurations associated with the instability of the system in the configuration of a regular octahedron. The general splitting pattern, the wave functions of the inversion levels, and the magnitude of the inversion splitting were obtained. The last turned out to be of the order of magnitude of the quanta used in radiospectroscopy. Hence, it is reasonable to expect that inversion splitting should be an important consideration in electron paramagnetic resonance (EPR) problems and the like. In fact, the presence of a number of closely spaced levels, each of which for the case $S = 1/2$ is a Kramers doublet, should significantly alter the system of levels in a magnetic field compared to what would be expected for a single doublet with no inversion splitting.

On the other hand, not all experimental facts from EPR spectra in the systems considered find satisfactory explanations. Even the most simple and most studied (see [2]) example of the spectrum of Cu^{2+} remains unclear. In particular, a satisfactory interpretation has not yet been given for the temperature transition from one type of spectrum to another (see below) that is observed in some of its compounds. Hence, the investigation of the effect of inversion splitting on EPR spectra

began naturally with octahedral complexes of the Cu^{2+} ion. It was expected that taking inversion splitting into account would lead to a satisfactory explanation of the most important experimental peculiarities of the spectra of these systems, in particular, the temperature dependence. As will be seen below, these expectations were fulfilled.

The present paper sets forth the general method of obtaining the system of levels of octahedral complexes of transition metals in a magnetic field, with account taken of inversion splitting and spin-orbit interaction, and applies it to the case of Cu^{2+} . The characteristics of the set of spin-inversion levels obtained can be important for the establishment of a system with a large negative effective temperature.

2. INVERSION LEVELS WITH SPIN-ORBIT INTERACTION

In the octahedral Cu^{2+} complex there are two inversion levels^[1]: the ground level A_{1g} and an excited, doubly degenerate level E_g . The wave functions corresponding to them have the form¹⁾:

$$\begin{aligned}\Psi_{1x} &= [3(1 - \gamma_x)]^{-1/2} (\Phi_{1x} + \Phi_{2x} + \Phi_{3x}), \\ \Psi_{2x} &= [3(2 + \gamma_x)]^{-1/2} (2\Phi_{1x} - \Phi_{2x} - \Phi_{3x}), \\ \Psi_{3x} &= [2 + \gamma_x]^{-1/2} (\Phi_{2x} - \Phi_{3x});\end{aligned}\quad (1)$$

¹⁾These functions differ from those in [1] by the specification of normalization constants.

$$\Phi_{ix} = \psi_i(q) \chi_{ix}, \quad \chi_{ix} = \prod_{\alpha=1}^p \chi_{n_\alpha}(Q_\alpha^{(i)}), \quad i = 1, 2, 3, \\ \gamma_\kappa = \int \chi_{ix} \chi_{j\kappa} dQ_\alpha, \quad i \neq j. \quad (2)$$

Here $\psi_i(q)$ is the electronic function of the missing 3d electron in the 3d¹⁰ shell, $\chi_{n_\alpha}(Q_\alpha^{(i)})$ is the wave function of the n_α -th excited state of the harmonic oscillator corresponding to the normal coordinate Q_α in the i -th equilibrium configuration, and κ is shorthand for the set of quantum numbers $n_1, \dots, n_p$ characterizing the vibrational state of the complex.

The electronic functions in the approximation considered are expressed by the usual hydrogen d-functions φ_4 and φ_5 , the angular parts of which are respectively $3z^2 - r^2$ and $x^2 - y^2$ in Cartesian coordinates. For the case of tetragonal distortion in the equilibrium configurations consisting of an elongation along one axis of the octahedron,

$$\psi_1 = \varphi_5, \quad \psi_2 = -\frac{1}{2}\varphi_5 + \frac{\sqrt{3}}{2}\varphi_4, \quad \psi_3 = -\frac{1}{2}\varphi_5 - \frac{\sqrt{3}}{2}\varphi_4. \quad (3)$$

For shortened octahedra the ψ_i are similar and can be obtained from (3) by exchanging φ_4 and φ_5 . The expression for the magnitude of the inversion splitting and the numerical values for certain cases were given in [1].

Each of the inversion states (1) is a spin doublet [$S = 1/2$; the states corresponding to projections of the spin on the Oz axis are symbolized by (+) and (-)]. These states are not coupled by the spin-orbit interaction, but are subjected to the influence of the nearest excited state T_{2g} , described by the electronic d-functions [1] φ_1 , φ_2 , and φ_3 , the angular parts of which equal respectively xy , xz , and yz . Choosing the expression for the spin-orbit interaction in the form $\lambda \hat{L} \cdot \hat{S}$ where λ is the spin-orbit coupling constant, and symbolizing the separation between the terms E_g and T_{2g} by Δ , it is possible to obtain the following expressions for the wave functions of the Kramers doublets of Cu^{2+} ion in a free octahedral complex by first-order perturbation theory (equilibrium configurations—elongated octahedra):

$$\Psi_{1\kappa}^{(\pm)'} = \Psi_{1\kappa}^{(\pm)} + \frac{\lambda}{\Delta} [3(1 - \gamma_\kappa)]^{-1/2} \times \left\{ -i \left(\chi_{3\kappa} - \frac{1}{2} \chi_{2\kappa} - \frac{1}{2} \chi_{1\kappa} \right) \varphi_3^{(\mp)} \left(\chi_{2\kappa} - \frac{1}{2} \chi_{1\kappa} - \frac{1}{2} \chi_{3\kappa} \right) \right. \\ \left. \times \varphi_2^{(\mp)} \mp i \left(\chi_{1\kappa} - \frac{1}{2} \chi_{2\kappa} - \frac{1}{2} \chi_{3\kappa} \right) \varphi_1^{(\pm)} \right\}, \\ \Psi_{2\kappa}^{(\pm)'} = \Psi_{2\kappa}^{(\pm)} + \frac{\lambda}{\Delta} [3(2 + \gamma_\kappa)]^{-1/2} \left\{ i \left(\chi_{1\kappa} + \chi_{3\kappa} - \frac{1}{2} \chi_{2\kappa} \right) \varphi_3^{(\mp)} \right. \\ \left. \mp \left(\chi_{1\kappa} + \chi_{2\kappa} - \frac{1}{2} \chi_{3\kappa} \right) \varphi_2^{(\mp)} \right. \\ \left. \mp i \left(2\chi_{1\kappa} + \frac{1}{2} \chi_{2\kappa} + \frac{1}{2} \chi_{3\kappa} \right) \varphi_1^{(\pm)} \right\},$$

$$\Psi_{3\kappa}^{(\pm)'} = \Psi_{3\kappa}^{(\pm)} + \frac{\lambda}{\Delta} [4(2 + \gamma_\kappa)]^{-1/2} \left\{ i (\chi_{3\kappa} + 2\chi_{2\kappa}) \varphi_3^{(\mp)} \right. \\ \left. \pm (2\chi_{2\kappa} + \chi_{3\kappa}) \varphi_2^{(\mp)} \pm i (\chi_{2\kappa} - \chi_{3\kappa}) \varphi_1^{(\pm)} \right\}. \quad (4)$$

Similar expressions can be obtained for the case when the equilibrium octahedra are tetragonally flattened out (shortened along a diagonal).

3. SPLITTING IN A MAGNETIC FIELD; g FACTORS

Since the splitting in an external magnetic field can turn out to be of the same order as the inversion splitting, its effect should be calculated at the same time as the interaction between the equivalent configurations of the complex U. [1] The complete perturbation of these configurations therefore has the form

$$U + \lambda(\hat{L}, \hat{S}) + \beta(\hat{L} + 2\hat{S}, \mathbf{H}). \quad (5)$$

The matrix elements of this expression are determined rather easily if we start with the functions (4), in which basis the matrix $U + \lambda \hat{L} \cdot \hat{S}$ is diagonal. We obtain a sixth-order secular equation to find the energy levels $\epsilon_{i\kappa}$. Its solution for arbitrary direction of the field is difficult. We shall first consider the direction $\mathbf{H} \parallel \text{Oz}$. For this we find

$$\epsilon_{1\kappa}^\pm = -\frac{1}{6} \delta_\kappa \pm \left[1 - \frac{\lambda}{\Delta} \left(\frac{5}{2} + \frac{3}{4} \gamma_\kappa \right) \right] \beta H \\ - \frac{1}{2} \left\{ \left[\delta_\kappa \mp \frac{\lambda}{\Delta} \left(1 + \frac{3}{2} \gamma_\kappa \right) \beta H \right]^2 + \frac{8\lambda^2}{\Delta^2} \left(1 - \frac{3}{4} \gamma_\kappa \right)^2 \beta^2 H^2 \right\}^{1/2}, \\ \epsilon_{2\kappa}^\pm = -\frac{1}{6} \delta_\kappa \pm \left[1 - \frac{\lambda}{\Delta} \left(\frac{5}{2} + \frac{3}{4} \gamma_\kappa \right) \right] \beta H \\ + \frac{1}{2} \left\{ \left[\delta_\kappa \mp \frac{\lambda}{\Delta} \left(1 + \frac{3}{2} \gamma_\kappa \right) \beta H \right]^2 + \frac{8\lambda^2}{\Delta^2} \left(1 - \frac{3}{4} \gamma_\kappa \right)^2 \beta^2 H^2 \right\}^{1/2}, \\ \epsilon_{3\kappa}^\pm = \frac{1}{3} \delta_\kappa \pm \left[1 - \frac{\lambda}{\Delta} \left(1 - \frac{3}{2} \gamma_\kappa \right) \right] \beta H. \quad (6)$$

The dependence of these levels on magnetic field intensity is shown in the figure. We see that as a consequence of the interaction of the levels 1^+ and 2^+ , 1^- and 2^- , two regions of differing behavior of the levels in the magnetic field arise, separated by a certain interval (indicated in the figure by the dashed lines) in which there is a mixing of the levels. The position of this interval depends on the ratio between δ_κ and βH . In the region to the right of it $\lambda \beta H / \Delta \gg \delta_\kappa$; this we call the high-frequency region for short. There is significant probability for three transitions here, as indicated in the figure by arrows. The g factors corresponding to these transitions are

$$g_1 = 2 - \frac{8\lambda}{\Delta},$$

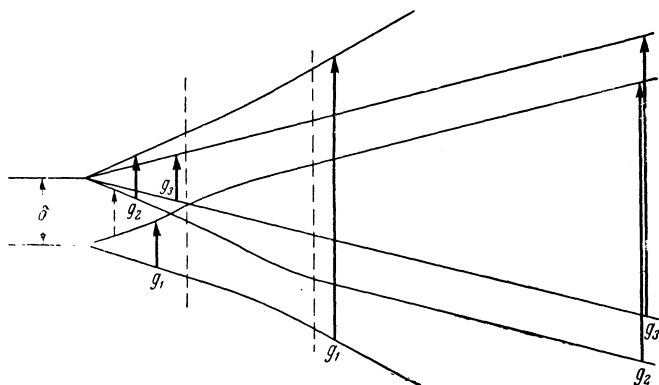
$$g_2 = 2 - \frac{\lambda}{\Delta} (2 + 3\gamma_\kappa), \quad g_3 = 2 - \frac{\lambda}{\Delta} (2 - 3\gamma_\kappa). \quad (7)$$

In the low-frequency region, where $\lambda\beta H/\Delta \ll \delta_K$, three other transitions are allowed, with g factors:

$$g_1 = 2 - \frac{4\lambda}{\Delta},$$

$$g_2 = 2 - \frac{\lambda}{\Delta} (6 + 3\gamma_K), \quad g_3 = 2 - \frac{\lambda}{\Delta} (2 - 3\gamma_K). \quad (8)$$

In the intermediate region the probabilities of the low-frequency transitions decrease with increasing field and become vanishingly small at its right edge. The high-frequency transitions disappear at the left boundary.



For directions of H along the Ox or Oy axes the energy level spectrum is the same as (6). However, in intermediate directions, not along the coordinate axes (the tetragonal axes of the octahedron), the levels approach each other in such a way that the differences in the g factors decrease, and in the trigonal axis direction all g values are the same and equal to $g = 2 - 4\lambda/\Delta$, both in the low-frequency and also in the high-frequency spectrum.

The general picture of the high-frequency spectrum is as if there were three paramagnetic ions absorbing at the same time, each situated in a nearly tetragonal field (the terms containing γ_K are in fact second-order corrections, with the directions of the tetragonal axes of these ions mutually orthogonal. In the low-frequency spectrum there is one completely isotropic transition g_1 , corresponding to the ground inversion state, and two anisotropic transitions located symmetrical on both sides of the isotropic line. The latter are at their maximum separation from the isotropic transition when the magnetic field is along the tetragonal axes of the octahedron and are closest to it when the field is along the trigonal axes.

In the case when the equilibrium configurations of the complex are (flattened) octahedra that are shortened along a diagonal, the spectral pattern is

in principle similar to that obtained above; however, instead of (7), we have for the g factors of the high-frequency spectrum:

$$g_1 = 2, \quad g_2 = 2 - \frac{\lambda}{\Delta} (6 + 3\gamma_K), \quad g_3 = 2 - \frac{\lambda}{\Delta} (6 - 3\gamma_K). \quad (9)$$

4. TEMPERATURE DEPENDENCE OF THE SPECTRUM. EXPLANATION OF THE TEMPERATURE TRANSITION OBSERVED IN THE CRYSTALLINE STATE.

An interesting peculiarity of this spectrum is its temperature dependence. This shows up in the magnitude of the inversion splitting δ_K , which is determined by the vibrational state κ and grows exponentially with temperature (even for a free complex). Temperature dependence via the parameter γ_K is less effective, since it enters only in terms of the second order, as has already been noted.

As δ increases, the transition region of the spectrum shifts to the right (see the figure), i.e., the upper limit of the low-frequency spectrum and the lower limit of the high-frequency spectrum become higher. For a constant (and sufficiently high) frequency of absorbed electromagnetic radiation, the following temperature dependence should be observed. At low temperatures, the spectrum corresponds to the high-frequency system of lines, given by the g factors of (7) [or (9)]. As the temperature is increased, a point corresponding to the right boundary of the transition region is reached; here the intensity of the high-frequency spectrum begins to fall off and the low-frequency lines begin to appear. With further increase of temperature, the left boundary is reached; the spectrum is the low-frequency spectrum and is determined by the g factors of (8).

It is obvious that the direct observation of this temperature dependence is possible only in complexes within crystals. In fact, in the liquid and gaseous states the observed spectrum is averaged over all directions. The complication in the spectrum brought about by the inversion splitting is manifested only indirectly, in the line width and its temperature and frequency dependence. The investigation of this question will be the subject of a separate communication.

In order to observe the spectrum described above in the crystalline state, at least two conditions must be fulfilled: (1) The symmetry of the surroundings of the given complex must not remove the E_g degeneracy of its electronic ground state (this will stabilize one of the three equivalent

Experimental data on the temperature transition in the EPR spectrum of the $\text{Cu}(\text{H}_2\text{O})_6^{2+}$ complex in salts of trigonal symmetry

Crystal	Region of the temperature transition, ° K	Low-temperature spectrum			High-temperature spectrum, g_1
		g_1	g_2	g_3	
(Cu,Zn) $\text{SiF}_6 \cdot 6\text{H}_2\text{O}$	~12—50	2.46	2.10	2.10	2.22
(Cu,Zn) $(\text{BrO}_3)_2 \cdot 6\text{H}_2\text{O}$	~7—35	—	—	—	2.22
(Cu,Mg) $_3\text{La}_2(\text{NO}_3)_{12} \cdot 24\text{H}_2\text{O}$	33—45	2.47	2.10	2.10	2.22
(Cu,Mg) $_3\text{Bi}_2(\text{NO}_3)_{12} \cdot 24\text{H}_2\text{O}$	~20—90	2.45	2.10	2.10	2.22
$\text{Cu}_3\text{La}_2(\text{NO}_3)_{12} \cdot 24\text{H}_2\text{O}$	~170—270	2.41	2.10	2.10	2.22

configurations of the tetragonally distorted octahedron), i.e., must not remove the inversion splitting, and (2) the effect of these surroundings must be sufficiently small, so that the distortion of the octahedron in the equilibrium configuration and the height of the barrier between these configurations remain sufficiently large; otherwise, the criterion of applicability of the theory of inversion splittings [1] is not fulfilled.

It is proper to mention at this point that in a crystal the height of the barrier between equivalent equilibrium configurations of a complex really depends also on the state of interaction of its ligands with the external surroundings (on their mobility) and, consequently, depends strongly on temperature (on the excitation of vibrations of the crystal as a whole). Hence the inversion splitting δ should be expected to be a strong function of the temperature independently of the population of the vibrational states of the complex.²⁾

The hydrated salts of copper with trigonal symmetry best fulfill the requirements set forth above. In these salts the ionic sites are occupied by the hydrated complex $\text{Cu}(\text{H}_2\text{O})_6^{2+}$, and the surrounding field, having trigonal symmetry, does not remove the inversion splitting in this complex. It is necessary to compare our results with the EPR measurements on these compounds.

Some experimental EPR data [3-6] on a group of similar crystals are presented in the table. In all cases a temperature transition from one type of spectrum to another is observed. Thorough studies [5,6] have shown that at low temperatures the observed spectrum corresponds to the presence of three types of Cu^{2+} ions in the crystal, each of which is situated in a tetragonal field, with the three tetragonal axes mutually perpendicular and forming the edges of a cube (or the diagonals of an

octahedron), the trigonal axis of which coincides with the crystal axis. As the temperature is raised, the intensity of this spectrum begins at a certain point to diminish to zero, and, at the same time, a new type of spectrum consisting of a single isotropic line begins to appear with increasing intensity. The limits of the region of the temperature transition are indicated approximately in the table. The g factors corresponding to the field directed along the tetragonal axis of one of the Cu^{2+} ions are given for the low-temperature spectrum.

It is easy to see that the general picture presented here of the observed spectrum and its temperature dependence is in good agreement with the theory. The g factors of the low-temperature spectrum correspond to Eq. (7), which pertains to elongated (along a diagonal) octahedra in the equilibrium configurations (this again confirms that elongated octahedra are more stable than shortened ones, see [7]). In the intermediate region the correspondence between theory and experiment is also complete. The isotropic line with $g = 2.22$ observed in the high-temperature spectrum corresponds to a transition between the sublevels of the ground inversion state with $g_1 = 2 - 4\lambda/\Delta$. Even such a detail as a certain diminution in the value of g_1 with an increase in temperature near the upper limit of the transition region [$g = 2.235 \pm 0.005$ at $T = 45^\circ\text{K}$ and $g = 2.218 \pm 0.003$ at $T = 90^\circ\text{K}$ for $(\text{Cu}, \text{Mg})_3\text{La}_2(\text{NO}_3)_{12} \cdot 24\text{H}_2\text{O}$ [6]] also follows directly from the temperature dependence of the energy levels in a magnetic field given in (6) (see the figure; an increase in temperature corresponds to a shift along the spectrum from right to left). However, transitions corresponding to excited inversion levels in the high-temperature spectrum have as yet not been clearly observed. It is known only that some additional weak lines whose origin is unclear have been observed. [3] It is possible that more careful experimental investigation will permit a positive identification of

²⁾Any kind of accurate calculations of the barrier height and its temperature dependence are scarcely possible or convenient here, since they involve the complete calculation for the crystal as a whole.

these transitions.³⁾ One can expect to observe them more easily at low temperatures and respectively lower frequencies.

Thus, the temperature transition observed in these systems is naturally explained within the framework of the inversion splitting theory. Likewise it is clear that this transition does not have its origin in the properties of the crystal as a whole and in particular in any kind of phase transition, as has been proposed by Ingram,^[8] but is a property of the octahedral complex. The entire spectrum arises from one complex ion, and the presence of three unequivalent ions in the crystal at low temperatures is not required for its explanation. The crystalline state, by stabilizing the directions of the axes of the octahedral complex in space, only makes the observation of the spectrum easier.⁴⁾

It also follows from the explanation presented that the temperature transition from one spectrum to the other should be higher, the higher the frequency of the absorbed electromagnetic radiation.

5. FREQUENCY DEPENDENCE OF THE SPECTRUM. POSSIBLE ABSORPTION IN ZERO FIELD.

Another interesting property of the investigated spectrum is its frequency dependence. The character of the dependence of the spectrum on the frequency of the electromagnetic radiation at constant temperature is similar to the temperature dependence at constant frequency with the only difference that the changes in the spectral pattern with increasing frequency (moving from left to right in the figure) correspond to the changes in the spectrum with diminishing temperature. A detailed description of the expected low-frequency and high-frequency spectra and their changes in the intermediate region was given above. At low frequencies one can hope to see the g_2 and g_3

lines, which cannot be distinguished clearly in the high-temperature spectrum. Experiments to study the frequency dependence of the spectrum in the aspect considered above have not been carried out, to our knowledge.

Besides the usual allowed transitions of which we are speaking, there is in this spectrum also the possibility of transitions the matrix elements of which arise only from excited orbital states and the spin-orbit interaction and are of order λ/Δ in the magnetic dipole approximation. This also includes zero-field transitions between inversion levels. The magnitude of the absorption quantum corresponding to these transitions equals the inversion splitting δ , which can be determined approximately by comparing the theoretical data on the temperature dependence of the energy levels [from Eqs. (6)] with the experimental data (results of approximate calculations of δ are in ^[1]). Knowing the size of the quantum being used $h\nu = g\beta H$, it suffices to determine that value of δ , in relation to which $\lambda\beta H/\Delta$ in (6) becomes negligibly small, that corresponds to the temperature of the left boundary of the transition interval. Similarly, the value of δ for the right boundary of this interval can be found. For $(\text{Cu, Mg})_3\text{La}_2(\text{NO}_3)_{12} \cdot 24\text{H}_2\text{O}$, on which detailed measurements have been made,^[6] it is easy to find that for $T = 33^\circ\text{K}$, $\delta \sim 100$ Mc/sec, and for $T = 45^\circ\text{K}$, $\delta \sim 100$ Mc/sec, and for $T = 45^\circ\text{K}$, $\delta \sim 10^5$ Mc/sec. Approximately the same values of δ for other crystals will be attained at other temperatures, as can be seen from the table. In view of the strong temperature dependence of δ the corresponding absorption is best sought at constant frequency by means of varying the temperature.

Other transitions with approximately the same probability could be 1^+-2^- , 1^+-3^- , 1^--2^+ , 1^--3^+ in the low-frequency spectrum (the transition 1^+-2^- is indicated by the broken arrow in the figure) and analogously in the high-frequency spectrum. A characteristic property of these further transitions is that, in contradistinction to all the other transitions, they do not disappear (more correctly, they do not combine with the main lines) for field direction along the trigonal axis of the octahedron. Just such lines were observed by Bleaney and Ingram.^[3] For their further identification it is necessary to keep in mind that along the trigonal axes the lines 1^+-2^- and 1^+-3^- come together and are at a distance $H_1 - H_2 = (1/g_1\beta)(\delta - 2h\nu)$ from the main line g_1 , where $h\nu$ is the magnitude of the absorbed quantum. Knowing the position of the additional line and the frequency, one can estimate the magnitude of the inversion splitting from

³⁾It is possible that the difficulty in observing these lines is above all due to the fact that at high temperatures δ is large; this can lead to a violation of the criterion of applicability of the theory of inversion splitting,^[1] and, consequently, to the disappearance of the excited inversion levels. It is also probable that the width of these levels is large on account of relaxation transitions to lower inversion states not associated with a change in spin state.

⁴⁾It should be noted that, in principle, correct qualitative proposals as to the origin of the temperature transition in the spectrum were in fact made earlier^[9,10] (see also ^[11]), but they could not be regarded as convincing. An explanation for the isotropic low-temperature line in the spectrum was first proposed by Abragam and Pryce.^[12]

this relation.⁵⁾ It is also easy to see that the position of this line, as well as the frequency, depends also on the temperature.

Further experimental tests of the results of this paper would be of great interest.

In conclusion, the author expresses his sincere thanks to A. M. Prokhorov for valuable discussions.

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ON THE PROPERTIES OF DISPLACEMENTS IN p -SPACE OF CONSTANT CURVATURE

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The displacements in constant-curvature space, which are among the basic elements of the generalized field theory developed by the author^[2], are studied in detail. A characteristic of the displacements, the focusing property, is revealed, having no analogue in the ordinary Euclidean case. The field theory apparatus of^[2] is refined. Deviations from the energy-momentum conservation law in scattering processes are considered; it is shown that the law remains valid in the generalized theory for the elastic scattering of fermions. The focusing property results in an interaction anomaly in a particle-antiparticle system. The possible formation of bound states of such systems during contact interactions with a small coupling constant is discussed briefly.

1. INTRODUCTION

IN earlier work^[1,2] the present author constructed a field theory in which the pseudo-Euclidean momentum space was replaced with a p -space of constant curvature. In this theory the role of the ordinary addition of momenta is played by the displacement operator of p -space. In the present work the properties of the displacements are studied in detail. As in^[2], we shall first consider an elliptic p -space having a positive-definite metric.¹⁾ Certain relations for displacements in the physical region (pseudo-elliptic space) will be considered in Sec. 6.

In investigating the system of displacements as a whole their parameterization by means of the vector k (k -parameterization) is not entirely suitable. The displacement equation in the case of k -parameterization is^[2]

$$q = d_0(k)p = \frac{p\sqrt{1+k^2} + k[1 - pk/(1 + \sqrt{1+k^2})]}{1 - pk} \quad (1.1)$$

containing $\sqrt{1+k^2}$, which must undergo a change of signs as k^2 varies continuously.²⁾ Elliptic p -space is a closed "one-sided space."^[3] "Infinitely distant" points of this space forming a three-dimensional hyperplane are not isolated points and go over into "finite" points under displacements. The one-sidedness of the space consists in the fact the two infinitely distant points of a single ray are

considered identical. More precisely, for each function of the vector $p = \beta n$ ($n^2 = 1$) we must obtain identical values when $\beta \rightarrow +\infty$ and $\beta \rightarrow -\infty$. This property will hereinafter be called the projective invariance of the theory. It is easily shown that (1.1) will be projectively invariant only if the sign of $\sqrt{1+k^2}$ is reversed when k^2 passes through infinity. The indeterminate sign of $\sqrt{1+k^2}$ is reversed when k^2 passes through infinity. The indeterminate sign of $\sqrt{1+k^2}$ creates certain inconveniences when the displacement equation (1.1) is used. In order to overcome this difficulty we introduce another, rational, parameterization of the displacement system (l -parameterization).

The displacement operation (1.1) satisfies the correspondence principle by going over into ordinary addition for small p and k . However, for large momenta the properties of displacements differ extremely from those of ordinary momentum addition. Specifically, when $q = -p$ we have the so-called focusing property. The present work consists essentially in an investigation of this property. From the physical point of view the focusing property signifies the appearance of an anomaly in the interaction between a particle and an antiparticle having identical momenta (Sec. 7).

2. l -PARAMETERIZATION OF DISPLACEMENTS

With each vector l of p -space we associate a displacement defined by

$$q = d_0(l)p = l + \frac{1+l^2}{1-l^2-2lp}(l+p) \quad (2.1)$$

This equation is a rational function of the vectors p and l , and satisfies the condition of projective

¹⁾The notation of^[2] will be used unless specified otherwise.

²⁾I. E. Tamm pointed out the need to consider both signs of $\sqrt{1+k^2}$, without which the displacement equation (1.1) becomes incomplete.

invariance, as is easily shown. The advantage of (2.1) over (1.1) lies in the fact that the vector q is determined uniquely in terms of p and l . The parameterization of displacements represented by (2.1) will be called l -parameterization.

The relation between the two displacement formulas is given by

$$k = 2l/(1 - l^2). \quad (2.2)$$

Substituting (2.2) in (1.1), we obtain (2.1). In virtue of (2.2) some vector k corresponds uniquely to each vector l . Fixing k and solving (2.2) for l , we obtain two values (positive roots):

$$l_1 = \frac{k}{1 + \sqrt{1 + k^2}}, \quad l_2 = \frac{k}{1 - \sqrt{1 + k^2}}. \quad (2.3)$$

For the scalar product of the vectors in (2.3) we have $l_1 l_2 = -1$, and therefore the distance [2]

$$s(l_1, l_2) = \pi/2. \quad (2.4)$$

The vector k given by (2.2) is by definition the momentum transferred to a fermion at the vertex of a Feynman diagram. Because of the two values in (2.3) each momentum k corresponds to two essentially different displacements $d_0(l_1)$ and $d_0(l_2)$.

Let us consider the limiting case $k \rightarrow 0$. It is convenient to write $k = \beta n$, where $\beta \rightarrow 0$ and n is an arbitrary unit vector ($n^2 = 1$). From (2.3) we obtain

$$l_1 \approx \frac{1}{2} \beta n = \frac{1}{2} k \rightarrow 0, \quad l_2 \approx -2n/\beta. \quad (2.5)$$

Substituting (2.5) in (2.1), we obtain expressions for the displacements (with a fixed vector p):

$$q_1 = d_0(l_1) p \approx \beta n + p = k + p, \quad (2.6)$$

$$q_2 = d_0(l_2) p \approx -p + 2(n p) n.$$

The considered case of extremely small momenta k illustrates clearly the essential difference between the displacements $d_0(l_1)$ and $d_0(l_2)$. While the first displacement is of classical character (satisfying the correspondence principle), the second shift is the product of an inversion relative to a hyperplane perpendicular to the vector n and the simultaneous reflection of all four axes. The solution l_2 in (2.5) and the corresponding displacement $d_0(l_2)$ in (2.6), which have no analogues in the ordinary theory, will be called nonclassical. The nonclassical displacement $d_0(l_2)$ is an improper rotation of elliptic p -space (its determinant is -1). Kadyshvskii [3] has discussed the possibility of continuous transitions from proper rotations to improper rotations in one-sided spaces of constant curvature.

The introduction of the l -parameterization of displacements (2.1) enables us to refine the meaning of the correspondence principle in the theory being developed. According to (2.5) and (2.6) correspondence is not achieved for small values of the transferred momentum k , but rather for small vectors of the vector l . The latter therefore acquires physical meaning, while k , which is given by (2.2), becomes a secondary quantity in a sense. The "classical region" in which the correspondence principle is satisfied will pertain to values $l^2 \ll 1$. Equation (2.5) shows that small momenta k also correspond to "nonclassical" values of l ($l^2 \gg 1$). The contribution of nonclassical displacements to real physical processes is considered in Sec. 5.

For a more detailed investigation of (2.1) it is convenient to employ a geometric construction. Having in mind the definition of the displacement operator for a scalar function of the vector p ,

$$\hat{d}_0(l) f(p) = f(d_0(-l)p), \quad (2.7)$$

we shall [in distinction from (2.1)] use q to denote the vector

$$q = d_0(-l)p = -l + \frac{1 + l^2}{1 - l^2 + 2lp}(p - l) = \frac{1 + l^2}{1 - l^2 + 2lp} p - \frac{2(1 + lp)}{1 - l^2 + 2lp} l. \quad (2.8)$$

The sum of the coefficients of the vectors $(-p)$ and $(-l)$ in the right-hand side of (2.8) equals unity. It follows that the three points $(-p)$, $(-l)$, and q lie on the same straight line (Fig. 1). From (2.8) we obtain

$$1 - ql = \frac{1 + l^2}{1 - l^2 + 2lp}(1 + pl), \quad (2.9)$$

$$1 + q^2 = \left(\frac{1 + l^2}{1 - l^2 + 2lp} \right)^2 (1 + p^2). \quad (2.10)$$

Combining (2.9) and (2.10), we obtain

$$|1 - ql| \sqrt{1 + q^2} = |1 + pl| \sqrt{1 + p^2}. \quad (2.11)$$

From Eq. (1.2) in [2], representing the distance between two points of elliptic p -space, and (2.11) we obtain the following relations for the distance between the given points:

$$s(p, l) \equiv s(-p, -l) = s(-l, q). \quad (2.12)$$

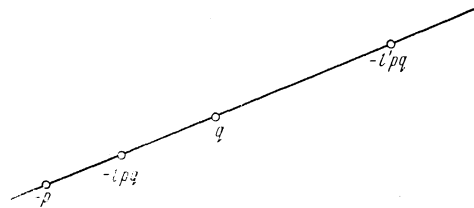


FIG. 1

The geometric interpretation of displacements is based on (2.12).

In investigating (2.12) we must take into account the facts that: 1) a straight line in elliptic space is closed (being equivalent topologically to a circle) with its total length equal to π , and 2) by definition the length of a segment of a straight line does not exceed $\pi/2$.

We introduce the abbreviated notation

$$\lambda = s(p, l) = s(-l, q), \quad \rho = s(-p, q). \quad (2.13)$$

We easily find, using Fig. 1,

$$\rho = 2\lambda \quad (0 \leq \lambda \leq \pi/4), \quad \rho = \pi - 2\lambda \quad (\pi/4 \leq \lambda \leq \pi/2). \quad (2.14)$$

The geometric construction (Fig. 1) enables us to establish graphically an extremely important property of displacements in elliptic space. Let us put

$$\lambda = s(p, l) = \pi/2. \quad (2.15)$$

Then, in virtue of (2.14), we have $\rho = s(-p, q) = 0$ and therefore $q = -p$, since the elliptic p-space has a positive-definite metric. The condition (2.15) is equivalent to the relation

$$pl = -1. \quad (2.16)$$

For a fixed value of p the vectors l satisfying (2.16) fill a three-dimensional hyperplane of p-space. We arrive at the conclusion that when p and l satisfy (2.15) [or, equivalently, (2.16)] the displaced vector

$$q = d_0(-l)p = -p \quad (2.17)$$

does not depend on the direction of the straight line passing through the points $(-p)$ and $(-l)$. This property of the displacements in elliptic space, for which there is no analogous property in the usual theory, will be called the focusing property; its physical interpretation will be considered in Sec. 7.

We shall now solve (2.8) for l with fixed values of p and q . In Fig. 1 the point $(-l)$ must lie on a straight line passing through the points $(-p)$ and q and must also satisfy (2.12). It is easily seen that two solutions exist:

$$l = l_{pq} \quad (0 \leq \lambda \leq \pi/4), \quad l = l'_{pq} \quad (\pi/4 \leq \lambda \leq \pi/2). \quad (2.18)$$

For $q = -p$ the construction no longer applies. In this case (2.8) has an infinite number of solutions; this corresponds to the focusing property. The solutions of (2.18) satisfy

$$s(l, l') = \pi/2. \quad (2.19)$$

In explicit analytic form the solutions of (2.18) are

$$l_{pq} = \frac{p\sqrt{1+q^2} - q\sqrt{1+p^2}}{\sqrt{1+q^2} + \sigma\sqrt{1+p^2}}, \quad l'_{pq} = \frac{p\sqrt{1+q^2} + q\sqrt{1+p^2}}{\sqrt{1+q^2} - \sigma\sqrt{1+p^2}}, \quad (2.20)$$

where

$$\sigma = \text{sign}(1 - pq) \quad (2.21)$$

determines the selection of the branch of $\sqrt{1+p^2}$.

When $q = -p$ an indeterminacy associated with the focusing property arises in (2.20). To investigate this indeterminacy we write $q = -p + \beta n$, where $\beta \rightarrow 0$ and $n^2 = 1$. In the limit $\beta = 0$, we obtain

$$l_{pq} = p, \quad l'_{pq} = p - \frac{1+p^2}{(\rho n)} n. \quad (2.22)$$

Through the l -parameterization of displacements in p-space we are enabled to simplify the expression for the displacement operator in the spinor representation. In view of (2.2) the spinor displacement operator is easily represented by

$$\hat{d}_s(l) = (1 - \hat{l})/\sqrt{1 + l^2}. \quad (2.23)$$

The operator $\hat{d}_s(l)$ acts on the constant spinor ψ . As in [2], we define the total displacement operator $\hat{d}(l)$ acting on a spinor function $\psi(p)$ as the product of the operators (2.7) and (2.23):

$$\hat{d}(l) = \hat{d}_0(l) \hat{d}_s(l). \quad (2.24)$$

3. MATRIX ELEMENTS OF DISPLACEMENT OPERATORS

We shall now calculate explicitly the matrix elements of a displacement operator in the p-representation. Since the spinor displacement operator $\hat{d}_s(l)$ is given explicitly by (2.23) as a matrix, it is sufficient to calculate the matrix elements of $\hat{d}_0(l)$. We shall first consider the more general problem of determining the matrix elements of the operator

$$\hat{B} = \int d\Omega_l b(l) \hat{d}_0(l), \quad (3.1)$$

where $b(l)$ is an arbitrary function of l . Letting $\hat{B}$ operate on $f(p)$, we obtain

$$\hat{B}f(p) = \int d\Omega_l b(l) f(q), \quad (3.2)$$

where q , which depends on p and l , is given by (2.8). In (3.2) integration over $d\Omega_l$ must be replaced by integration over $d\Omega_q$. In order to obtain the relation between the respective volume elements we write the expression for a volume element of p-space in polar coordinates:

$$d\Omega_q = \sin^3 \rho \sin^2 \eta \sin \theta d\rho d\eta d\theta, \quad (3.3)$$

where φ , θ , η , and ρ are the appropriate angle variables, which have the limits

$$0 \leq \varphi \leq 2\pi; \quad 0 \leq \theta, \eta \leq \pi; \quad 0 \leq \rho \leq \pi/2. \quad (3.3a)$$

We note that ρ represents geometrically the distance from some fixed point taken as the coordinate origin to an arbitrary point q of p -space. Using Fig. 1, taking the point $(-p)$ as the coordinate origin, and considering that the points $(-p)$, $(-l)$, and q lie on the same straight line and that $d\Omega_l = d\Omega_l$, we easily obtain the relation between the volume elements:

$$\begin{aligned} d\Omega_l &= d\Omega_q/16 \cos^3(\rho/2) & (0 \leq \lambda \leq \pi/4), \\ d\Omega_l &= d\Omega_q/16 \sin^3(\rho/2) & (\pi/4 \leq \lambda \leq \pi/2), \end{aligned} \quad (3.4)$$

where λ and ρ are defined by (2.13). Using (3.4) for the change of variables in (3.2), we obtain

$$\hat{B}f(p) = \int d\Omega_q \left\{ \frac{b(l_{pq})}{16 \cos^3(\rho/2)} + \frac{b(l'_{pq})}{16 \sin^3(\rho/2)} \right\} f(q), \quad (3.4a)$$

whence, from the definition of a matrix element,

$$\langle p | \hat{B} | q \rangle = \frac{b(l_{pq})}{16 \cos^3(\rho/2)} + \frac{b(l'_{pq})}{16 \sin^3(\rho/2)}. \quad (3.5)$$

The quantities l_{pq} and l'_{pq} are given by (2.20). From the definition (3.1) of the operator $\hat{B}$ we obtain directly

$$\hat{d}_0(l) = \delta \hat{B} / \delta b(l). \quad (3.6)$$

The variational derivative of (3.5) yields the matrix element

$$\langle p | \hat{d}_0(l) | q \rangle = \frac{1}{16} \left\{ \frac{\delta(l, l_{pq})}{\cos^3(\rho/2)} + \frac{\delta(l, l'_{pq})}{\sin^3(\rho/2)} \right\}. \quad (3.7)$$

Using (2.14), this can be written in the form

$$\langle p | \hat{d}_0(l) | q \rangle = \frac{1}{16 \cos^3 \lambda} \{ \delta(l, l_{pq}) + \delta(l, l'_{pq}) \}, \quad (3.8)$$

where $\lambda = s(p, l)$.

The amplitudes of different processes are expressed in terms of the matrix elements for products of the displacement operators. Let us consider the diagonal matrix element for the product of two displacements that represents a polarization operator in a second-order perturbation.

The following general relation is convenient. Let $\hat{A}$ be an operator with $\langle p | \hat{A} | q \rangle$ as its matrix element. From the definition (2.7) of the displacement operator we have directly

$$\begin{aligned} \langle p | \hat{d}_0(l) \hat{A} | q \rangle &= \langle d_0(-l)p | \hat{A} | q \rangle, \\ \langle p | \hat{A} \hat{d}_0(l) | q \rangle &= \langle p | \hat{A} | d_0(l)q \rangle. \end{aligned} \quad (3.9)$$

By relatively simple calculations we obtain

$$\langle p | \hat{d}_0(l) \hat{d}_0(-t) | p \rangle = \{ \delta(l, t) + \delta(l, t') \} / 16 \cos^3 \lambda. \quad (3.10)$$

Here l and t are arbitrary vectors of p -space,

$\lambda = s(p, l)$, and t' is a point lying on the straight line (p, l) and satisfying the condition $s(t, t') = \pi/2$. For $\lambda = \pi/2$, (3.10) has a singularity directly related to the focusing property of the displacement operation. This singularity results in divergence of the integral of (3.10) over $d\Omega_p$. Methods of eliminating this difficulty are discussed in the following section. It must be noted, however, that the divergence considered here has nothing in common with the divergences encountered in the usual field theory. This divergence is of extremely general mathematical character associated with the fact that the trace of any operator with a continuous spectrum is always infinite.

4. NORMALIZED DISPLACEMENTS

The appearance of a nonintegrable singularity in (3.10) leads to serious difficulties in calculating the matrix elements for diagrams containing closed loops. This situation is directly related to the existence of the focusing property in the displacement operators of p -space. This focusing property is an intrinsic property of elliptic p -space and from our point of view plays an extremely essential role in the construction of a physical theory of elementary particles (see Sec. 7). We shall therefore consider one of the possible ways of eliminating the indicated difficulty.

We suppose that the displacement operator (2.24) is not a good analogue of momentum addition in the theory based on a p -space of constant curvature. We introduce instead of (2.24) the so-called normalized displacement as an analogue of momentum addition. It must be emphasized that the introduction of a normalized displacement in the theory is a special hypothesis, the correctness of which can only be determined by the subsequent development of this theory.

We define the normalization operator $\hat{N}(l)$, which is diagonal in the p -representation, by its matrix element

$$\langle p | \hat{N}(l) | q \rangle = \cos^{3/2} s(p, l) \delta(p, q). \quad (4.1)$$

$\hat{N}(l)$ contains as its parameter a vector l of p -space, and satisfies the relation

$$\hat{N}(l) \hat{d}_0(l) = \hat{d}_0(l) \hat{N}(-l). \quad (4.2)$$

The validity of (4.2) follows from the fact that in virtue of (3.8) the matrix element $\langle p | \hat{d}_0(l) | q \rangle$ differs from zero only for $l = l_{pq}$ and $l = l'_{pq}$, and that for these values of l we have $s(p, l) = s(q, -l)$ according to (2.12). Therefore

$$\begin{aligned} \langle p | \hat{N}(l) \hat{d}_0(l) | q \rangle &= \cos^{3/2} s(p, l) \langle p | \hat{d}_0(l) | q \rangle \\ &= \langle \hat{d}_0(l) | q \rangle \cos^{3/2} s(q, -l) = \langle p | \hat{d}_0(l) \hat{N}(-l) | q \rangle, \end{aligned} \quad (4.2a)$$

which is equivalent to (4.2).

We define the normalized displacement operator for a scalar function by

$$\hat{d}_{N0}(l) = \hat{N}(l) \hat{d}_0(l). \quad (4.3)$$

The normalized displacement for a spinor field $\psi(p)$ is defined as a product:

$$\hat{d}_N(l) = \hat{d}_{N0}(l) \hat{d}_s(l) = \hat{N}(l) \hat{d}_0(l) \hat{d}_s(l). \quad (4.4)$$

The normalized displacement hypothesis signifies that the operator $\hat{d}_N(l)$ is taken to be a suitable analogue of momentum addition at vertices of Feynman diagrams. It is obvious that $\hat{d}_N(l)$, like the displacement operator $\hat{d}(l)$, satisfies all requirements of the correspondence principle in the classical region. Using the unitarity of the displacement operator (2.24) and the relation (4.2), we easily obtain the adjoint operator of (4.4):

$$\hat{d}_N^+(l) = \hat{d}_N(-l). \quad (4.5)$$

When in (3.10) the displacement $\hat{d}_0(l)$ is replaced with the normalized displacement $\hat{d}_{N0}(l)$, we obtain

$$\begin{aligned} \langle p | \hat{d}_{N0}(l) \hat{d}_{N0}(-l) | p \rangle &= \frac{1}{16} \{ \delta(l, t) \\ &+ \text{ctg}^{3/2} s(p, t) \delta(l, t') \}. \end{aligned} \quad (4.6)^*$$

The matrix element (4.6) no longer contains a non-integrable singularity. The same will evidently apply to all higher perturbation approximations. Thus the introduction of the normalized displacement operator (4.4) eliminates the difficulty mentioned at the end of the preceding section.

5. REFINEMENT OF THE FIELD THEORY

By introducing the l -parameterization of displacements along with the normalized displacement operator $\hat{d}_N$, we achieve a refinement of the field theory apparatus constructed in [2] (Sec. 2). The required changes reduce to a redefinition of the operator for a boson quasi-field. We define this operator by

$$\varphi(l) = (\mu^2/4 + l^2)^{-1/2} \{ c(l) + c^*(-l) \}, \quad (5.1)$$

where $c(l)$ and $c^*(l)$ are the particle annihilation and creation operators, which are related by the commutators

$$[c(l), c^*(t)] = \delta(l, t). \quad (5.2)$$

(Commutators equal to zero are not written out.)

From (5.1) and (5.2) we obtain for the average of $\varphi(l)\varphi(t)$ over the vacuum

$$\langle \varphi(l)\varphi(t) \rangle_0 = (\mu^2/4 + l^2)^{-1} \delta(l, -t). \quad (5.3)$$

The boson propagation function in the right-hand side of (5.3) was selected so that for small transferred momenta k the contribution from the non-classical solution I_2 would, in virtue of (2.5), be negligibly small compared with that of the classical solution I_1 . This agrees with the refined correspondence principle formulated in Sec. 2.

The operator $\hat{\varphi}$ generalizing the field $\varphi(x)$ is defined by

$$\hat{\varphi} = \frac{1}{2\pi^2} \int d\Omega_l \varphi(l) \hat{d}_N(l). \quad (5.4)$$

The normalization coefficient of this integral was selected to satisfy the correspondence principle in making a transition to the ordinary theory. We use $\hat{\varphi}$ to define the interaction operator

$$\Lambda = g \langle \bar{\Psi} | \gamma_s \hat{\varphi} | \Psi \rangle, \quad (5.5)$$

which in virtue of (4.5) is Hermitian:

$$\Lambda = \Lambda^+. \quad (5.5a)$$

Beginning at this point, all further construction of the theory is exactly the same as in [2]. For a direct four-fermion interaction (5.5) is replaced by the operator

$$\Lambda = \int d\Omega_l J(l) J(-l), \quad (5.6)$$

where the current operator is

$$J(l) = \langle \bar{\Psi} | \hat{\partial} \hat{d}_N(l) | \Psi \rangle. \quad (5.6a)$$

6. DISPLACEMENT OPERATIONS IN THE PHYSICAL REGION

In this section we shall consider certain relations for displacements in a pseudo-elliptic space that generalizes the ordinary pseudo-Euclidean p -space and possesses a non-definite metric. All 4-momentum vectors mentioned in this section belong to the pseudo-elliptic space. The scalar product of two vectors in this space is written as $pq = p_0q_0 - \mathbf{p} \cdot \mathbf{q}$. The relationship between the elliptic and pseudo-elliptic spaces has been considered in [2]. A displacement in pseudo-elliptic space with l -parameterization is given by

$$q = d_0(l)p = l + \frac{1-l^2}{1+l^2+2lp}(l+p). \quad (6.1)$$

The vector l is related to the transferred momentum k by

$$k = 2l/(1+l^2) \quad (6.2)$$

*ctg = cot.

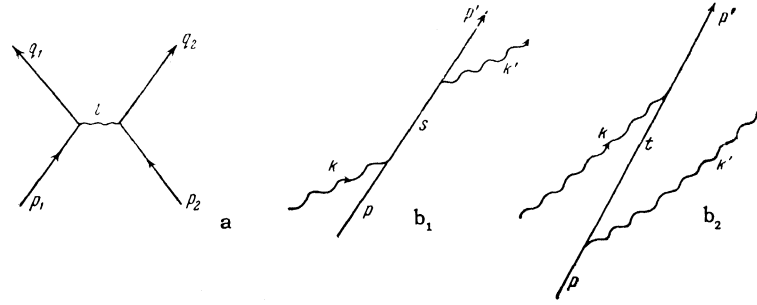


FIG. 2

As in Sec. 2, we obtain two solutions of (6.2):

$$l_1 = k/(1 + \sqrt{1 - k^2}), \quad l_2 = k/(1 - \sqrt{1 - k^2}). \quad (6.3)$$

For a real physical particle l must [according to (5.3)] satisfy $l^2 = \mu^2/4 \ll 1$.³⁾ Therefore the solution l_2 drops out for real particles and we obtain a unique correspondence between the vectors k and l , with l always belonging to the classical region.

Using the simplest scattering processes as examples, we shall now consider to what extent energy and momentum are not conserved when ordinary p-space is replaced by a space of constant curvature.

1. Fermion-fermion scattering (Fig. 2a). The initial and final momenta are related by

$$q_1 = d_0(l) p_1, \quad q_2 = d_0(-l) p_2. \quad (6.4)$$

Since all particles participating in the reaction are real, their momenta satisfy the conditions

$$p_i^2 = m_i^2, \quad q_i^2 = M_i^2 \quad (i = 1, 2), \quad (6.5)$$

where all masses are different in general. We use a relation analogous to (2.20) for the case of the pseudo-elliptic space. We can limit ourselves to the classical solution l_{pq} , since the nonclassical solution l'_{pq} either leads to infinitely large values of l , which make a vanishingly small contribution, or, if l is finite, they must satisfy

$$q = -p. \quad (6.6)$$

Equation (6.6), representing the focusing property, can be satisfied only for processes involving the creation or annihilation of pairs of particles. The contribution of the focusing property to the annihilation diagrams is discussed in the following section. Here we shall limit ourselves to scatter-

ing processes not involving pairs. From (6.4) and (6.5) we have the following relation between momenta:

$$\frac{q_1 / \sqrt{1 - M_1^2} - p_1 / \sqrt{1 - m_1^2}}{1 / \sqrt{1 - M_1^2} + 1 / \sqrt{1 - m_1^2}} = \frac{p_2 / \sqrt{1 - m_2^2} - q_2 / \sqrt{1 - M_2^2}}{1 / \sqrt{1 - m_2^2} + 1 / \sqrt{1 - M_2^2}}. \quad (6.7)$$

Equation (6.7) shows that the energy-momentum conservation law is not of universal character in the theory under discussion. The relation between the momenta of the considered particles depends on their masses. In its usual form the energy-momentum conservation law is satisfied, as a general rule, up to m^2 (in dimensionless units). For elastic scattering ($m_1 = M_1 = m$, $m_2 = M_2 = m'$, since $m = m'$ is not obligatory) Eq. (6.7) is simplified and leads to the usual conservation law

$$p_1 + p_2 = q_1 + q_2. \quad (6.8)$$

Equation (6.7) was obtained by considering the simplest second-order diagram. It must be emphasized, however, that this relation is of purely kinematic character and is based on the definition (6.1) of displacements. For higher-order corrections the situation is complicated by the fact that different displacements do not commute. In virtue of certain considerations discussed in [2] effects associated with mass smearing arise; this in turn requires a fundamentally new approach to the description of a state of real particles. These questions require further investigation; however, we can assume that the final results will not be essentially different.

2. Boson-fermion scattering (Fig. 2b). For the sake of simplicity we shall consider the Compton effect: $k^2 = 0$. In this case we have, from (6.2), $l = k/2$. Momenta are related by

$$s = d_0(k/2) p = d_0(k'/2) p_1 \quad (\text{diagram } b_1), \quad (6.9)$$

$$t = d_0(-k'/2) p = d_0(-k/2) p' \quad (\text{diagram } b_2).$$

The momenta of the initial and final fermions satisfy

$$p^2 = p'^2 = m^2. \quad (6.10)$$

³⁾The relation $l^2 = \mu^2$ is not satisfied rigorously in the theory being discussed because of "mass smearing" resulting from the noncommutativity of displacements (Sec. 5 of [2]). However, we shall here neglect the small smearing effect and shall consider these relations as applying to real particles.

Eqs. (6.9) and (6.10) yield the following result. Let $\mathcal{P} = p + k$ be the energy-momentum vector of the system before scattering, and let $\mathcal{P}' = p' + k'$ be the corresponding vector after scattering. We conserve the relativistic invariant

$$E_0^2 = \mathcal{P}^2 = \mathcal{P}'^2. \quad (6.11)$$

E_0 is the total energy in the center-of-mass system. An essential difference of the present case from the usual theory consists in the fact that the center-of-mass system before scattering does not coincide with the center-of-mass system after scattering. The relative velocity of these two reference systems differs for the two diagrams b_1 and b_2 in Fig. 2 and is a second-order small quantity with regard to certain relativistic invariants formed from the vectors p , k , and k' .

7. ANOMALOUS PARTICLE-ANTIPARTICLE INTERACTIONS

A special case of $q = -p$ (6.6) which was excluded from the preceding section corresponds to the focusing property for nonclassical displacements (Sec. 2). Eq. (6.6) can be satisfied only for particle-antiparticle scattering where the two particles have equal momenta; by Feynman's rules the amplitude contains the antiparticle momentum with the opposite sign. It can therefore be expected that in the theory being developed here specific singularities will appear in expressions describing particle-antiparticle interactions in states with equal (or nearly equal) momenta.

On the other hand, the contribution of the focusing property to the interaction of two identical particles is extremely small and leads to no interaction anomaly. This sharp qualitative difference between particle-particle and particle-antiparticle interactions (which like the focusing property itself has no analogue in the ordinary theory) is characteristic of the field theory in a p -space of constant curvature.

We shall therefore consider briefly the formation of bound particle-antiparticle states, which play a fundamental role in the construction of composite elementary particle models.^[4-7] In contrast to the kinematic analysis in Sec. 6 the present question is of dynamical character. The quantitative description of the pertinent effects is extremely sensitive to the selected type of particle interaction. Therefore for the purpose of an exact quantitative calculation we must, first of all, refine the form of the interaction operator Λ . This is not included among our present tasks, and we shall confine ourselves to a qualitative analysis, using the simplest four-fermion interaction type

(5.6) as an example, and selecting the operator Λ in the form

$$\Lambda = g \int \langle \bar{\psi} | \gamma_5 \hat{d}_N(l) | \psi \rangle \langle \bar{\psi} | \gamma_5 \hat{d}_N(-l) | \psi \rangle d\Omega_l. \quad (7.1)$$

In contrast to Sec. 6 the particle momenta do not obey the "reality" condition $p^2 = m^2$; therefore our subsequent examination will be performed in the elliptic space. Using (4.3), (4.4), and (3.8), we easily obtain the matrix element of a normalized displacement:

$$\langle p | \hat{d}_{N0}(l) | q \rangle = \{ \delta(l, l_{pq}) + \delta(l, l'_{pq}) \} / 16 (\cos \lambda)^{1/2}. \quad (7.2)$$

The singularity of the matrix element at $\lambda = \pi/2$, which corresponds to the focusing property of displacements, enables us to simplify the determination of particle-antiparticle scattering amplitudes. We shall consider in first approximation the amplitude corresponding to a sum of annihilation diagrams (Fig. 3). It can be shown that in this approximation singularities of the two-particle Green's function are determined by the singularities of the matrix element $\langle l | \hat{\Pi} | t \rangle$, where the operator $\hat{\Pi}$ is related to the polarization operator $\hat{P}$ by

$$\hat{\Pi} = (1 + g\hat{P})^{-1}. \quad (7.3)$$

In (7.3), $\hat{P}$ denotes the polarization operator corresponding to the simplest fermion loop. In accordance with the general rules in^[2], its matrix element is

$$\langle l | \hat{P} | t \rangle = \text{Tr} \{ (m + \hat{p})^{-1} \gamma_5 \hat{d}_N(l) (m + \hat{p})^{-1} \gamma_5 \hat{d}_N(-t) \}. \quad (7.4)$$

This matrix element is based largely on (4.6), from which it follows that the operator contains a nondiagonal part [the second term in (4.6)]. The nondiagonal part of $\hat{P}$ leads to "smearing" of the meson mass, thus greatly complicating the analysis. Neglecting the smearing effect, we shall consider only the diagonal part of $\hat{P}$ associated with the first term of (4.6).

Omitting the details of the calculation,⁴⁾ we

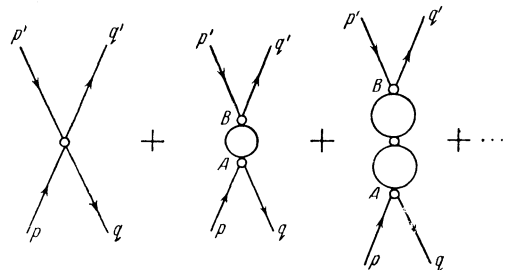


FIG. 3

⁴⁾The procedure for calculating (7.4), the generalization of Feynman parametrization, and a more consistent solution of the present problem will be treated in a separate paper.

present the final form of the diagonal matrix element:

$$\langle l | \hat{P}_D | t \rangle = \pi^2 \delta(l, t) [\Phi(\cos^2 \xi) - \Phi(\sin^2 \xi)], \quad (7.5)$$

where we have the function

$$\Phi(z) = z \left\{ -\frac{1}{3} + \frac{1}{2} \int_0^1 \frac{dx}{1-4x(1-x)z} \left(\frac{1}{\sqrt{1-m^2} \sqrt{1-4x(1-x)z}} \right. \right. \\ \left. \left. \times \ln \frac{1+\sqrt{1-m^2} \sqrt{1-4x(1-x)z}}{1-\sqrt{1-m^2} \sqrt{1-4x(1-x)z}} - 2 \right) \right\}, \quad (7.6)$$

and the quantity ξ is related to l by

$$l^2 = \operatorname{tg}^2 \xi. \quad (7.7)^*$$

Because of the singularity of the matrix element (7.2) at the vertices A and B of the diagrams in Fig. 3, we can consider (7.5) only for the values $l = l'_{pq}$, with $q = -p$, given by (2.22). From the latter we obtain

$$\cos^2 \xi = \frac{1}{1+l^2} = \frac{(pn)^2}{(1+p^2)(1+p^2-(pn)^2)} = \frac{p^2 \cos^2 \alpha}{1+p^2}, \quad (7.8)$$

where α is the angle between the vectors p and n that is measured at the point p in the metric of the elliptic p-space.

The total momentum of the system is

$$\mathcal{P} = p - q = 2p. \quad (7.9)$$

Using μ to denote the mass of a bound state ($\mathcal{P}^2 = -\mu^2$) and neglecting μ^2 compared with unity, we obtain from (7.8) and (7.9)

$$\cos^2 \xi = -\frac{1}{4} \mu^2 \cos^2 \alpha. \quad (7.10)$$

For the existence of a bound state it is required that for some value of μ the expression

$$\langle \{1 + g\pi^2 [\Phi(-\frac{1}{4}\mu^2 \cos^2 \alpha) - \Phi(1 + \frac{1}{4}\mu^2 \cos^2 \alpha)]\}^{-1} \rangle \quad (7.11)$$

should exhibit a singularity. The angular brackets in (7.11) denote an average over α [over the directions of the vector n in (2.22)]. The explicit form of this average can be determined only through a complete solution of the problem. For the purpose of an estimate we can use the fact that as μ^2 increases the denominator in (7.11) vanishes for the first time when $\cos^2 \alpha = 1$. This gives the following equation for μ^2 :

$$1 + g\pi^2 [\Phi(-\mu^2/4) - \Phi(1 + \mu^2/4)] = 0. \quad (7.12)$$

In virtue of $\mu^2 \ll 1$, it can be shown that (7.12) will have a solution for

$$g \approx (\pi^2 [\Phi(1) - \Phi(0)])^{-1} \approx 6/\pi^2. \quad (7.13)$$

The coupling constant g in (7.1) is related to the usual four-fermion interaction constant G by

$G = \pi^4 g$. The estimate (7.13) in dimensionless units therefore gives $G \approx 6\pi^2$. By Kadyshevskii's hypothesis^[3] we have $G_F \approx 1$ for the coupling constant of weak interactions. Therefore the interaction (7.1) is only 60 times stronger than the universal Fermi interaction, and is thus very weak.

Several authors (in^[8], for example) have attempted to calculate bound states of a particle-antiparticle system for a contact interaction within the framework of the ordinary theory. In our view the results of these calculations are accidental to a considerable extent, because they necessarily introduce an arbitrary cutoff of divergent integrals. It is even more significant that the ordinary theory includes no intrinsic requirement of a sharp difference between the characters of particle-particle and particle-antiparticle interactions at small distances.

The situation is entirely different for the field theory in the p-space of constant curvature. First, this theory leads to no divergent expressions, because there is no arbitrariness in the results of the calculation. Secondly, the anomaly in the particle-antiparticle interaction follows from the intrinsic properties of the constant-curvature p-space. In conclusion, it must again be emphasized that the present analysis is only tentative, and that the qualitative results must be confirmed by detailed calculations.

8. CONCLUDING REMARKS

1. It has been shown in Sec. 6 that in the theory developed here the law of energy-momentum conservation is only approximate. However, in the extremely important case of elastic scattering this law is satisfied exactly [Eq. (6.8)] or in any event in the first nonvanishing approximation. An important consequence of general character follows therefrom for the theory of macroscopic bodies. Since in the theory of such bodies the only elementary interactions between particles involve Coulomb and gravitational forces, it can be expected that this theory will not be essentially changed. This eliminates to a considerable degree the general objections raised against the field theory in constant-curvature p-space which are based on the difficulties in treating macroscopic bodies within this theory.

2. The conclusions in Sec. 7 regarding the possibility of forming bound states in a baryon-antibaryon (meson) system with a small coupling constant provide a basis for expecting that the newly developed theory will require no special mechanism of strong interactions. We thus have the attractive prospect of basing the theory of elementary particles solely on universal weak and

* $\operatorname{tg} = \tan$.

electromagnetic interactions. In this picture strong interactions will be "secondary" effects associated with the formation of mesons as composite particles.

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THE RESISTANCE OF THIN PLATES AND WIRES IN A MAGNETIC FIELD

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The resistance of plates and wires that are thin compared with the mean free path, l , is calculated for the entire range of magnetic fields. In strong magnetic fields ($r \ll l$, where r is the Larmor orbit radius) the static "skin-effect" previously pointed out^[1] is taken into account.

1. INTRODUCTION

A large amount of work has been devoted to calculating the resistance of metals in a constant magnetic field H . The great interest in this question is associated with the possibility of additional checks on the basic assumptions of the modern electronic theory of metals, and of clarifying the features of the dispersion law for conduction electrons close to the bounding Fermi surface. The study of thin plates and wires can naturally throw additional light on this problem.

In the existing works, however, the range of fields H for which $r > d$ (r is the Larmor orbit radius, d is the thickness of the specimen; for $d \sim 10^{-3}$ cm this corresponds to $H < 10^4$ Oe) has been little studied. Also, the region of "strong" fields, $r < d$, has been treated without taking into account the inhomogeneity of the Hall field and the "screening" of the static current thereby caused. As shown in ^[1], this usually leads to incorrect results precisely in the case that allows the subsequent theory to be developed—the case of a specimen with a "good" surface (imperfections small compared with r), when the surface can be taken into account with an appropriate boundary condition. If the specimen surface is "bad" (imperfections of order r) the development of the subsequent theory is, on the one hand, extremely complex, and on the other hand is unnecessary, since all the results of interest to us can be obtained from simple physical considerations.

In the present paper the resistance of plates and wires is found in the entire range of magnetic fields for $d \ll l$ (l is the free path of electrons). It is assumed that the linear dimensions of the wire cross section are of equal order in all directions; the shape of the cross section is arbitrary. The direction of the magnetic field is everywhere taken as the z axis, and the direction of the total

current J is taken as the y axis (for the wire the y axis coincides of course with its axis).

2. STRONG MAGNETIC FIELD; $r \ll d$

A. If the magnetic field is very strong, $r \ll d$, and is not directed along the wire axis (or parallel to the plate surface) all the electrons playing an essential part collide with the surfaces, if $d \ll l$, so that d assumes the role of free path, and in all other respects everything proceeds just as in the bulk metal where $d \gg l$.

The resistance ρ of bulk metal for a specimen with a "poor" ¹⁾ surface has been found in a paper by I. Lifshitz, the present author, and Kaganov ^[2], $\rho_{\infty}^b = \rho(d, l)$, and for a "good" surface in a paper by the present author ^[1], $\rho_{\infty}^g = \rho(d, r, l)$ (for $d \ll l$ the condition for the applicability of the formulae of ^[1] is obviously satisfied). In accordance with what has been said we obtain for a thin specimen

$$\rho_d^b = \rho(r, d), \quad \rho_d^g = \rho(d, r, d). \quad (1)$$

Using (1) and the results of the works mentioned above^[1,2] we arrive at the following relations for the resistance:

Specimen with Bad Surface (Specimen Shape Arbitrary)

For $n_1 \neq n_2$ (n_1 and n_2 are respectively the number of electrons and "holes") we have

$$\rho_d^b(H) \sim \rho_{\infty}(0) l/d \sim d^{-1}, \quad (2)$$

and for $n_1 = n_2$

¹⁾The characteristic dimension δ of the random imperfections of the specimen surface is of course small compared with d . Therefore a "good" surface implies a magnetic field such that $r \gg \delta$, and a "poor" one has $r \ll \delta$ and occurs for thin specimens only in limiting strong magnetic fields.

$$\rho_d^b(H) \sim \rho_\infty(0) (l/d) (d/r)^2 \sim \rho_\infty(0) ld/r^2 \sim dr^{-2} \sim H^2 d, \quad (3)$$

where $\rho_\infty(0)$ is the resistance of an infinite specimen with $H=0$.

Specimen with Good Surface

For a wire with any ratio of n_1 to n_2 , for a plate with $n_1 \neq n_2$ and with $\mathbf{H}$ not lying in the $\xi\nu$ plane (ξ is the normal to the surface), and for a plate with $n_1 = n_2$ and with the magnetic field lying in the same plane as the total current $\mathbf{J}$ and the normal to the surface ξ , we have

$$\rho_d^g(H) \sim \rho_\infty(0) l/r \sim H. \quad (4)$$

For a plate with $n_1 = n_2$ in the remaining cases, and with $n_1 \neq n_2$ when $\mathbf{H}$ lies in the $\xi\nu$ plane but is not parallel to ξ we have

$$\rho_d^g(H) \sim \rho_\infty(0) ld/r^2 \sim d/r^2 \sim H^2 d. \quad (5)$$

For a plate with $n_1 \neq n_2$ and $\mathbf{H} \parallel \xi$ we have

$$\rho_d^g(H) \sim \rho_\infty(0) l/d \sim d^{-1}. \quad (5a)$$

B. We now consider the remaining special case in which the magnetic field is parallel to the axis of a wire (and consequently parallel to the direction of the total current) or to the surface of a plate. In this case the only characteristic parameter in directions normal to the surface is r , and the surface affects matters only at such distances. Therefore, in the case of a specimen with a poor surface (in the initial approximation the formulae for bulk metal are valid) the relative increase above the resistance of a bulk specimen is of order r/d , and is positive both for $n_1 \neq n_2$ with any mutual orientations of $\mathbf{J}$ and $\mathbf{H}$, and for $\mathbf{J} \parallel \mathbf{H}$ with any $n_1 - n_2$ (since the surface decreases the free path of the electrons colliding with it):

$$\rho_d^b(H) \sim \rho_\infty^b(H) (1 + \alpha r/d), \quad \alpha \sim 1,$$

i.e., for a wire and plate with $\mathbf{H} \parallel \mathbf{J}$, $n_1 \neq n_2$

$$\rho_d^b(H) \sim \rho_\infty(0) (1 + \alpha r/d) \quad (6)$$

and for a plate with $n_1 = n_2$ and $\mathbf{H}$ not parallel to $\mathbf{J}$

$$\rho_d^b(H) \sim \rho_\infty(0) (l/r)^2. \quad (7)$$

If the specimen surface is good, we obtain, using directly the formulae in [1]: for a wire with any $n_1 - n_2$

$$\rho_d^g(H) \sim \rho_\infty(0) (1 + \alpha r/d); \quad (8)$$

for a plate with $n_1 \neq n_2$ and $\mathbf{H} \parallel \nu$, or for $n_1 \neq n_2$ and any mutual orientations of $\mathbf{H}$ and ν

$$\rho_d^g(H) \sim \rho_\infty(0) (1 + \alpha r/d) \quad (9)$$

and for $n_1 = n_2$ and $\mathbf{H}$ not parallel to ν

$$\rho_d^g(H) \sim \rho_\infty(0) d/r \sim d/lr. \quad (10)$$

We point out the formulae obtained in other works for the particular case of a quadratic dispersion law. These are: the formulae for the plate with a good surface: 1) in a perpendicular field ($\mathbf{H} \parallel \xi$) [3,4] [formulae (5) and (5a) of the present paper], 2) in a parallel field with $n_1 \neq n_2$ [4-6] [formula (9) here].

It should be noted that for $r \ll d \ll l$ a curious phenomenon occurs—an oscillation of the resistance in an inclined magnetic field, which was first predicted by Sondheimer [3], and which was treated for an arbitrary dispersion law by Gurevich [7]. However, as noted in [1], Gurevich wrote down the conductivity tensor incorrectly, and—what is extremely important—did not take into account the inhomogeneity of the Hall current. The correction oscillatory in H can be found by successive approximations if the initial approximation found in [1] is used. The period of the oscillations then agrees, of course, with that found by Gurevich, but the order of magnitude of the amplitude can differ. The consideration of this question will be the subject of a separate communication.

3. WEAK MAGNETIC FIELD

We now proceed to fields corresponding to $r \gg d$. Here, of course, the quality of the surface does not play an important part. Because the smallest characteristic length is d , pronounced inhomogeneity of the electric field and current cannot occur in this depth. This allows us to write down immediately a formula for the effective conductivity and to obtain all the asymptotic relations of interest to us without solving the equation for the field normal to the surface (since it is clear at the start that $\mathbf{E}(\mathbf{r})$ is of the order of $\mathbf{E}$ at the surface), which makes the solution unusually simple compared with that we treated in [1]. Of course, because the actual form of $\mathbf{E}(\mathbf{r})$ is not found, the numerical constants in front of the corresponding expressions are not hereby determined. However, their evaluation presents no interest because they are related to the form of the collision integral, and to the way the electrons are reflected from the surface, and none of them can be used for comparing theory with experiment.

The calculation of the asymptotic relations presents no difficulties of principle; it is, however, an extremely tedious theoretical problem. Wherever possible, therefore, we shall find the form of $\rho(H)$ by using obvious physical considerations.

A. Weak Magnetic Field: $d \ll r$. Wire

In the case of a wire in a magnetic field inclined with respect to its axis, the field causes only a small "twisting" of the orbit, as a result of which the path traversed between collisions with the surface (since $d \gg l$, the mean free path does enter into the problem in the initial approximation with respect to the magnetic field) is of order $d \{1 + \alpha(d/r)^2\}$, $\alpha \sim 1$. Therefore,

$$\rho_d(H) \sim \rho_\infty(d) \left\{1 - \alpha \left(\frac{d}{r}\right)^2\right\} \sim \rho_\infty(0) \frac{l}{d} \left\{1 - \alpha \left(\frac{d}{r}\right)^2\right\}. \quad (11)$$

If the magnetic field is directed along the axis of the wire (and, consequently, parallel to the total current) electrons appear, which owing to the small radius of their orbits $r < d$ do not collide with the surface, and traverse without collision a path of order l . The relative number of such electrons corresponds to the relative number of orbits with diameters less than d when the mean diameter is of order $2r$. Since only electrons on the limiting Fermi-surface play an essential role ($r \sim cp_\perp/eH$, $p_\perp$ is the projection of electron momentum on the plane $p_x p_y$, $\mathbf{H} \parallel \mathbf{z}$), the electrons of interest to us correspond to the surface of a section of radius d on a surface of radius r , i.e., their relative number is of order $(d/r)^2$. The remaining electrons contribute a conductivity of the same order as in the absence of a magnetic field.

The resulting conductivity is thus

$$\sigma_d(H) \sim \sigma_d(0) + \alpha \sigma_\infty(0) \left(\frac{d}{r}\right)^2, \quad \alpha \sim 1$$

and the resistance is

$$\rho_d(H) \sim \rho_\infty(0) \left\{\frac{d}{l} + \alpha \left(\frac{d}{r}\right)^2\right\}^{-1} \sim \rho_d(0) \left\{1 + \alpha \frac{ld}{r^2}\right\}^{-1}, \quad \alpha \sim 1. \quad (12)$$

B. Weak Magnetic Field Parallel to the Surface; $r \gg d$. Plate

We now turn to the calculation of the resistance of a plate in fields such that $r \gg d$. It is apparent that this region can be divided into several ranges. In order to understand what influences the variation of ρ with $\mathbf{H}$ in such fields, we start with the case when the magnetic field is parallel to the surface of a plate.

The conductivity of a plate of thickness d in a magnetic field parallel to its surface is influenced by the following groups of electrons (see Fig. 1):

a) Electrons of type a, which do not collide with the surface and contribute a conductivity the same as in the bulk metal. The relative number of such



FIG. 1. The trajectories of electrons in a plate in a weak magnetic field parallel to its surface.

electrons is of order $(d/r)^2$ [see the derivation of formula (12)]. The conductivity associated with these electrons is

$$\sigma_{ik}^{(a)}(d, H) \sim (d/r)^2 \sigma_{ik}(\infty, H).$$

b) Electrons of type b, which collide with both surfaces of the plate. For $r \gg d$, these consist of almost all the electrons. These electrons are only slightly "twisted" by the magnetic field, and traverse a path $1 + \alpha(d/r)^2$ times greater than in zero magnetic field with $\alpha \sim 1$. In the absence of a magnetic field, the principal role is played by the "grazing" electrons that proceed almost parallel to the plate surface, (see [8,9]) which contribute a conductivity of order $(\sigma_0 d/l) \ln(l/d)$ (σ_0 is the conductivity of the bulk metal; $\sigma_0 d/l$ does not depend on l). In a magnetic field the maximum free path of electrons of type b is of order l if $\sqrt{rd} > l$, and of order $\sqrt{rd}$ if $\sqrt{rd} < l$ (since, if $\sqrt{rd} < l$, electrons reflected from the surface at an angle smaller than $\sqrt{rd}/d$ do not collide with the second surface, and the maximum path of electrons colliding with both surfaces is of order $\sqrt{rd}$). The contribution of these electrons to the conductivity is, therefore, of order

$$\sigma_0 \frac{d}{l} \ln \frac{l}{d} \text{ for } \sqrt{rd} > l, \quad \sigma_0 \frac{d}{l} \ln \frac{\sqrt{rd}}{d} \text{ for } \sqrt{rd} < l.$$

c) Electrons of type c that return to the same surface from which they set out. For $\sqrt{rd} < l$ the relative number of these electrons is of order $d/\sqrt{rd} \sim \sqrt{d/r}$ (since they consist of all electrons that avoid collisions with the second surface, i.e., those setting out at an angle $\beta \lesssim d/\sqrt{rd}$, and departure at any angle is almost equally probable), and the path traversed by these electrons is of order $\sqrt{rd}$, so that their contribution to the conductivity (which is proportional to the effective free path and the relative number of electrons) is of order

$$\sigma_0 \frac{\sqrt{rd}}{l} \sqrt{\frac{d}{r}} \sim \sigma_0 \frac{d}{l}$$

and does depend on the magnetic field.

For $\sqrt{rd} > l$ the majority of electrons of type c traverse a path of order l , do not have time to return to the surface without collisions in the bulk, and behave in the main just as in the absence of the magnetic field. There are, however, electrons with a small radius r' (i.e., with small p_y) such that $\sqrt{r'd} \lesssim l$. The relative number of such electrons

is of order $(r'/r)^2$, i.e., of order $(r_1/r)^2$ with $r_1 \sim l^2/d$. Of the similar electrons, those that depart at an angle $\beta \lesssim d/l$ (their relative number is d/l) are "safeguarded" by the magnetic field against colliding with the second surface, and traverse a path of order l , which is greater than in zero magnetic field. It is clear that the contribution of these electrons to the conductivity is of order $\sigma_0(r_1/r)^2 d/l$.

We now determine the conductivity of a plate in various magnetic fields (but with $r \gg d$).

1) Very weak magnetic fields: $r > r_1 \sim l^2/d \gg l$. Taking into account that here we have $\sigma_{ik}(\infty, H) \sim \sigma_{ik}(\infty, 0) \sim \sigma_0$, we easily see that in the initial approximation the conductivity is, as would be expected, the same as in zero magnetic field, and is provided by electrons of type b, while the principal field-dependent contribution to the conductivity is due to electrons of type c being "safeguarded" against collisions with the second surface. As a result:

$$\begin{aligned} \sigma(H, d) &\sim \sigma_0 \frac{d}{l} \ln \frac{l}{d} + \delta \sigma_0 \frac{d}{l} \left(\frac{r_1}{r} \right)^2 \\ &\sim \sigma(0, d) \left\{ 1 + \delta \frac{1}{\ln(l/d)} \left(\frac{r_1}{r} \right)^2 \right\}, \\ \delta &\sim 1, \quad r_1 \sim l^2/d. \end{aligned} \quad (13)$$

Thus even in fields such that $r \sim r_1 \sim l^2/d$ (for $d \sim 10^{-3}$ cm, $l \sim 10^{-1}$ cm, this corresponds to $H \sim 1$ Oe) a significant increase in the conductivity occurs (by a factor of about $1 + \delta \ln(l/d)$). It is clear from the method of derivation that a similar effect is only possible in a plate of length greater than l .

For a quadratic dispersion with $\mathbf{H} \parallel \mathbf{j}$ ($\mathbf{j}$ is the current density), formula (13) was first obtained in [6].

2) Weak magnetic fields: $l \ll r \ll r_1$. As before we have $\sigma_\infty(H) \sim \sigma_\infty(0) \sim \sigma_0$. The main contribution to the conductivity is due to electrons of type b, and

$$\sigma(d, H) \sim \sigma_0 \frac{d}{l} \ln \sqrt{\frac{r\alpha}{d}}, \quad \alpha \sim 1. \quad (14)$$

3) Strong magnetic fields: $d \ll r \ll l$. Electrons of types a and b play the principal roles. Because the conductivity components due to electrons of type a are strongly anisotropic, the entire effective conductivity tensor should be written out. Since d is the smallest length it is clear at the start that $E_y(y) \sim E_y(0)$ (see also the start of the section). Calculating the contribution these electrons make to the conductivity components in a way similar to that used in [1] (only now the electric field changes in a region of order of the orbit ra-

dus for the electrons of interest to us), we obtain

$$\begin{aligned} \hat{\sigma}_d(H) &\sim \sigma_\infty(0) \begin{pmatrix} k + \gamma\lambda^2 & k + \gamma\lambda^2 & k + \lambda^2 \\ k + \gamma\lambda^2 & k + \gamma^2\lambda^2 & k + \gamma\lambda^2 \\ k + \gamma\lambda^2 & k + \gamma\lambda^2 & k + \lambda^2 \end{pmatrix}; \\ \lambda &\sim \frac{r}{d}, \quad \gamma \sim \frac{r}{l}, \quad k \sim \frac{d}{l} \varepsilon \ln \frac{r\alpha}{d}, \quad \varepsilon = 1 - \frac{d}{2r_{\max}}, \end{aligned} \quad (15)$$

where λ^2 takes into account the relative number of electrons of type a, γ is the effect of the magnetic field on these electrons, $(d/l) \ln \sqrt{r\alpha}/d$ is the contribution to the conductivity of electrons of type b, and ε is the relative number of electrons of type b (it is obvious that when the maximum orbit diameter $2r_{\max}$ approximates to the thickness d of the plate, the number of electrons that collide with both surfaces diminishes; for $2r_{\max} < d$ there are in general no such electrons).

From (15) we easily find:

a) In the case when $1 - d/2r_{\max} \sim 1$ ($\varepsilon \sim 1$) i.e., when the field H is not close to the "critical" field (for which $2r_{\max} = d$), the effective resistivity tensor in the plane of the plate has the form:

$$\hat{\rho}_d(H) \sim \rho_\infty(0) \begin{pmatrix} k^{-1} & (k + \lambda^2)^{-1} \\ k^{-1} & (k + \lambda^2)^{-1} \end{pmatrix}. \quad (16)$$

For all directions of $\mathbf{H}$, except $\mathbf{H} \parallel \mathbf{J}$, we have

$$\rho_d(H) \sim \rho_\infty(0) \frac{l}{d} \left(\ln \frac{r\alpha}{d} \right)^{-1}. \quad (17)$$

If $\mathbf{H} \parallel \mathbf{J}$, then

$$\rho_d(H) \sim \rho_\infty(0) \frac{l}{d} \left\{ \ln \frac{r\alpha}{d} + \beta \frac{ld}{r^2} \right\}^{-1}, \quad \alpha, \beta \sim 1. \quad (18)$$

b) In the case $\varepsilon \ll 1$, when the field H is close to the "critical" field ($2r_{\max} \approx d$), the longitudinal specific resistance tensor has the form

$$\hat{\rho}_d(H) \sim \rho_\infty(0) \begin{pmatrix} 1 + k\gamma^{-2} & 1 \\ 1 + k\gamma^{-2} & 1 \end{pmatrix} \sim \rho_\infty(0) \begin{pmatrix} 1 + k^{-1}\varepsilon & 1 \\ 1 + k^{-1}\varepsilon & 1 \end{pmatrix}. \quad (19)$$

Thus for $\mathbf{H}$ not parallel to $\mathbf{J}$

$$\rho_d(H) \sim \rho_\infty(0) \left\{ 1 + \alpha \frac{l}{d} \left(1 - \frac{d}{2r_{\max}} \right) \right\}, \quad (20)$$

and for $\mathbf{H} \parallel \mathbf{J}$

$$\rho_d(H) \sim \rho_\infty(0). \quad (21)$$

C. Weak Inclined Magnetic Field, $r \gg d$. Plate

The trajectories of electrons in an inclined field (Fig. 2) differ from the trajectories in a parallel field in a way of interest to us only in that the electrons that do not collide with the surface should have not only a small radius, but also a small velocity v_z . In order that electrons should move

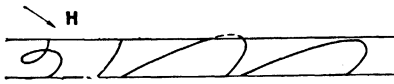


FIG. 2. The trajectories of electrons in a plate in an inclined magnetic field.

without collisions for a time τ , it is necessary that $v_z/v \sim d/l$. As a result, electrons of type a do not provide a significant contribution to the conductivity.

An exception is the particular case of the magnetic field normal to the surface, and electron orbits that are plane in real space when $v_z = 0$, i.e., v_z dependent only on p_z . The case of such special Fermi surfaces is completely analogous to the case of a quadratic dispersion law treated in [3,4], where the path along the film is of order r when $r < l$ and of order l when $r > l$; so that

$$\rho_d(H) \sim \rho_\infty(0) \frac{l}{d} \left\{ \ln \left(\frac{d}{l} + \frac{d}{r} \right) \right\}^{-1}.$$

In the general case for $r \gg l$ and in an inclined field we obtain formulae (13), (14). For $r \ll l$, because electrons of group a are unimportant, formula (14) is valid up to $r \sim d$.

CONCLUSIONS

We now present, using all the formulae obtained, graphs of the function $\rho(H)$ for the entire range of magnetic fields in the various cases: Fig. 3—the wire in an inclined field; Fig. 4—the wire in a longitudinal field; Fig. 5—the plate in an inclined field and Fig. 6—the plate in a parallel field [δ is the characteristic dimension of the random “roughnesses” of the surface (see [1]). The curves given allow the anisotropy of ρ to be seen easily.

It is curious to note: a) the rapid fall in the resistance of a plate at the field $r \sim l^2/d$ (for d

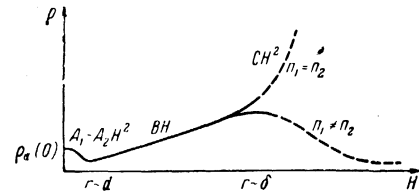


FIG. 3. The function $\rho(H)$ for a wire in an inclined field.

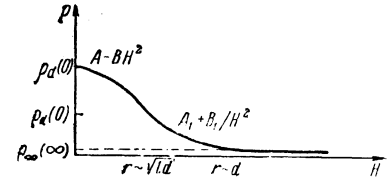


FIG. 4. The function $\rho(H)$ for a wire in a parallel field.

$\sim 10^{-3}$ cm, $l \sim 10^{-1}$ cm, this corresponds to $H \sim 1$ Oe); b) the linear relation that is the general case in strong fields and the break at $2r_{\max} = d$, for the plate, which allows the diameter of the Fermi surface to be determined; c) for $H \parallel J$ in Fig. 6, the slow change of resistance—the resistance increases for H not parallel to J by a factor of $\ln(l/d)$ on increasing the field by a factor of $(l/d)^2$ (curve $A_2 + B_2/\ln(H_0/H)$ and d) the rapid fall in resistance by a factor of l/d for only a severalfold increase in the field (curve $A_3 - B_3H$).

Also seen from the figures is the strong dependence of the resistance in high fields on the quality of the specimen surface (on the ratio of r to δ). We also note that the question of whether the curve going to saturation in Fig. 6 goes to $\rho_d(\infty)$ or differs from it depends, as is easily seen, on whether a direction of open trajectories coincides with the y -axis. The large number of characteristic points allows the quantities r , l , and δ to be determined for the same specimen under different conditions.

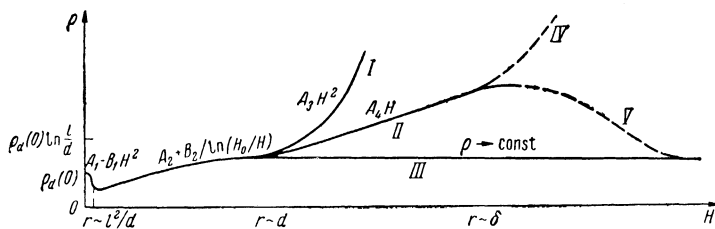


FIG. 5. The function $\rho(H)$ for a plate in an inclined field. Branch I is for $n_1 = n_2$, H not in the $\xi\nu$ -plane or $H \parallel \xi$; branch II for $n_1 = n_2$, H in the $\xi\nu$ -plane but $H \not\parallel \xi$, or for $n_1 \neq n_2$, H not in the $\xi\nu$ -plane; branch III for $n_1 \neq n_2$, H in the $\xi\nu$ -plane; branch IV, $A_3 H^2$, for $n_1 = n_2$; branch V for $n_1 \neq n_2$.

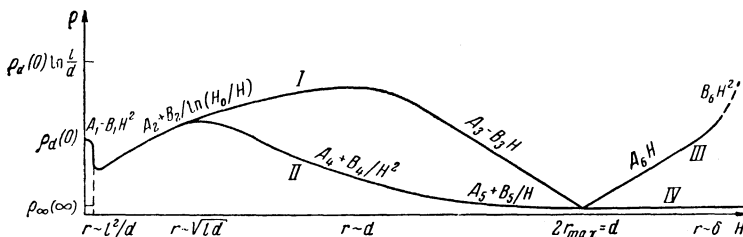


FIG. 6. The function $\rho(H)$ for a plate in a parallel field. Branch I is for $H \not\parallel \nu$; branch II for $H \parallel \nu$; branch III for $n_1 = n_2$, $H \not\parallel \nu$; branch IV for $n_1 \neq n_2$ or $n_2 = n_2$ and $H \parallel \nu$.

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OSCILLATIONS OF A WEAKLY INHOMOGENEOUS PLASMA

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The natural oscillations of a weakly inhomogeneous plasma are studied. The dispersion equations which determine the natural frequencies of such oscillations are similar to the quasi-classical Bohr quantization laws [see Eqs. (3), (55), (56)]. The high-frequency electron oscillations of a weakly inhomogeneous plasma are considered by means of such dispersion equations. The quantization rules are also used for the study of the oscillation spectrum of a weakly inhomogeneous plasma confined by a strong magnetic field. Sufficient conditions for instability of the plasma are deduced and in a number of concrete cases analyzed.

1. INTRODUCTION AND FUNDAMENTALS OF THE PROBLEM

RECENTLY the attention of many theoreticians who have been studying plasma physics has been drawn to the problem of the construction of a theory of electromagnetic waves in a weakly inhomogeneous plasma. (In particular, the necessity of such a theory is brought about by the urgent requirement for analysis of the question as to the stability of the plasma). Here attempts are made at an almost direct transfer of the methods applied in the description of a homogeneous plasma. Of course, such a procedure meets with a whole series of difficulties, among which can be noted, for example, the dependence of the natural frequencies of the vibrations on the coordinates.

On the other hand, it is well known that in the theory of propagation of electromagnetic waves in a weakly inhomogeneous medium, it has been possible to apply the method of geometric optics,^[1] which corresponds to the quasiclassical approximation of quantum mechanics. In the present report we set forth the application of such a method to the theory of the natural oscillations of a weakly inhomogeneous plasma.

For simplicity, let us consider the case of a one-dimensional dependence of the properties of the plasma on the single coordinate x . Then, by taking into account the dependence of the field on time and the other coordinates in the form $\exp(-i\omega t - ik_x y + ik_z z)$, one can reduce the field equation in a many cases to the form

$$\lambda^2 y'' + q(\omega, x)y = 0, \quad (1)$$

where λ is a small quantity, for example, of the order of the Debye or Larmor radius. When one says that λ is small, it is meant that λ is small in comparison with the characteristic length of change $q(x)$.

The asymptotic solutions^[2] of Eq. (1) corresponding to large λ have the form¹⁾

$$y = \frac{C}{(q(\omega, x))^{1/4}} \exp \left\{ \pm \frac{i}{\lambda} \int dx \sqrt{q(\omega, x)} \right\}. \quad (2)$$

The theory of the natural oscillations of a plasma is interested not only in the form of the characteristic solutions, but also in the frequency spectrum. Such a problem has been solved in principle for equations of type (1) by the construction of the quasi-classical approximation theory of quantum mechanics (see, for example, ^[2]). Here the eigenvalue spectrum is determined by the Bohr quantization conditions:

$$\int dx \sqrt{q(\omega, x)} = \pi n \lambda. \quad (3)$$

Here n is an integer that is large compared with unity.

A few words must be said about the limits of integration in the left-hand side of Eq. (3). If $q > 0$ throughout the entire plasma and the boundary condition of vanishing of the field on the boundary of the plasma is used, then the integral is taken over the entire plasma. If there is a single turning point at which $q > 0$, then the integral is taken from the turning point throughout the whole region $q < 0$. Finally, if there are several turning points then the

¹⁾An exception is the vicinity of the point where $q(x) = 0$.

integral over each region $q > 0$ will lead to a separate equation (3). The latter means that in the quasi-classical approximation the vibrations in each such region are independent. This is brought about by the fact that, as is known from quantum mechanics, the width of the quasi-classical barrier separating the classically achievable regions must be extremely large, and therefore the corresponding transmission coefficient is extremely small.

In quantum mechanics, $q(\omega, x)$ is always a real quantity. On the contrary, in the theory of natural oscillations of a plasma, q is frequently complex. The asymptotic solution of (2) has the same form for complex q . The situation concerning the quantization conditions (3), which give the dispersion equations of the plasma oscillations, is more complicated. A simplifying circumstance is the fact that the imaginary part of $q(\omega, x)$ is frequently small. This is always the case when the absorption of the plasma is small. Under these conditions, use can be made of dispersion equations of the type (3). Therefore, equations of the type (3) are applied below for the determination both of the real and imaginary parts of the frequencies of oscillation, and in particular for the study of the stability of the plasma.²⁾

Naturally, such consideration can be applied not only to the plasma oscillations described by Eq. (1), but everywhere where the natural oscillations of a weakly inhomogeneous plasma, which is described by a set of linear differential equations dependent on a small (or large) parameter, are involved.

2. HIGH FREQUENCY ELECTROMAGNETIC OSCILLATIONS

Following these general remarks, we now turn to a consideration of concrete examples. First let us consider high-frequency electronic oscillations of the plasma. In this case, we restrict ourselves to oscillations which are propagated in the direction of the inhomogeneity. Then one can write the following relation between the electric field and the displacement

$$D(x) = \left[1 - \frac{\omega_{Le}^2(x)}{\omega^2} \right] E(x) + 3 \frac{d^2}{dx^2} E r_D^2(x) \frac{\omega_{Le}^4(x)}{\omega^4}, \quad (4)$$

where

$$\omega_{Le}^2(x) = 4\pi e^2 N(x)/m, \quad r_D^2(x) = \kappa T(x)/4\pi e^2 N(x)$$

²⁾For complex q , the turning points lie in the complex plane x . Therefore, in the quantization rules (if, of course, they are possible) $\int q dx = 2\pi n \lambda$ the contour of integration must enclose such points in the complex plane (see [1]).

(this expression is valid under conditions for which the "wavelength" is large in comparison with the Debye radius). Equating the electric displacement to zero, we obtain an equation for oscillations of the longitudinal field which is obviously an equation of type (1). Therefore, the corresponding dispersion equation can be written in the form

$$\frac{1}{\sqrt{3}} \int \frac{dx}{r_D(x)} \frac{\omega^2}{\omega_{Le}^2(x)} \sqrt{1 - \frac{\omega_{Le}^2(x)}{\omega^2}} = \pi n, \quad (5)$$

Here the turning points are determined by the relation

$$\omega_{Le}^2(x) = \omega^2 \quad (6)$$

and correspond to a division of the plasma into regions of transparency and opacity.

In order to make the results obtained through the dispersion equation (5) more graphic and perceptible, let us first make a simplifying assumption on the constancy of the plasma pressure:

$$N(x) \kappa T(x) = \text{const} = N_0 \kappa T_0.$$

We further assume that the density of the plasma changes according to the law $N(x) = N_0 [1 + (x/d)^2]$. Then the dispersion equation (5) takes the following form:

$$\frac{1}{\sqrt{3}} \frac{d}{r_D(0)} \frac{|\omega|}{\omega_{Le}(0)} \left[\frac{\omega^2}{\omega_{Le}^2(0)} - 1 \right] = \pi n. \quad (7)$$

Inasmuch as the corresponding turning points are $\pm d[(\omega^2/\omega_{Le}^2(0)) - 1]^{1/2}$, then it is apparent that ω^2 must be larger than ω_{Le}^2 . Keeping this in mind, and also the fact that $r_D(0)/d$ is small in comparison with unity, we get from Eq. (7):

$$\omega^2 = \omega_{Le}^2(0) \{1 + 2n \sqrt{3} r_D(0)/d\}. \quad (8)$$

We note that in the case of a spatially homogeneous particle distribution in the region $-L/2 \leq x \leq +L/2$, the dispersion equation (5) leads to the following spectrum:

$$\omega^2 = \omega_{Le}^2(0) \{1 + 3(\pi n r_D(0)/L)^2\}. \quad (9)$$

3. LOW FREQUENCY PLASMA OSCILLATIONS IN A PLANE PLASMA LAYER CONFINED BY A MAGNETIC FIELD

This section is concerned with a plasma confined by a magnetic field. We shall speak only of the dependence of the plasma parameters on the inhomogeneity x . We assume the magnetic pressure to be considerably greater than the plasma pressure. Then we can establish that the magnetic field changes over distances that are much greater than the distances characteristic of change in the

particle distribution. Therefore, with great accuracy, we can neglect the effect of spatial change of the magnetic field on the motion of the plasma particles.

The case in which the Larmor radius of the particles can be regarded as small is a comparatively simple one. Then, by solving the kinetic equation with the self-consistent field, and by expanding the solution in powers of the Larmor radius, it is easy to obtain the following expression for the ionic charge density under the assumption that $\mathbf{E} = -\text{grad } \Phi$:

$$\frac{e^2 N_e}{m \Omega_e^2} \left[\frac{d^2 \Phi}{dx^2} + \frac{d\Phi}{dx} \frac{d \ln N}{dx} \right] - \Phi \frac{e^2}{m} \int_{-\infty}^{+\infty} \frac{v_z}{\omega - k_z v_z} \left\{ k_z \frac{\partial \tilde{f}_0}{\partial v_z} - \frac{k_y}{\Omega_e} \frac{\partial \tilde{f}_0}{\partial x} \right\}, \quad (10)$$

Here it is assumed that the plasma ions in the ground state are described by the distribution

$$f_0(v_z, v_\perp^2, x) + \frac{v_y}{\Omega} \frac{\partial f_0}{\partial x},$$

and $\tilde{f}_0 = \int dv_x dv_y f_0$. The z axis is directed parallel to the direction of the constant magnetic field B . In obtaining Eq. (10), it is also taken into account that ω and $k_z v_z$ are small in comparison with the Larmor frequency Ω . Moreover, it is assumed that $k_y R \ll 1$, where R is the Larmor radius. A similar expression holds also for electrons.

We limit ourselves to the case of values of the projection on the z axis of the phase velocity of the oscillations that are small in comparison with the thermal velocity of the electrons and large in comparison with the thermal velocity of the ions (or of sound for a non-isothermal plasma in which $T_e \gg T_i$; this is precisely the case considered below). Here the corresponding expansion can be introduced in the integrand of Eq. (10) both for electrons and ions. As a result, the following equation is obtained for the case of a Maxwellian electron distribution; this equation describes the potential of the electric field of the plasma oscillations:

$$\frac{d^2 \Phi}{dx^2} - k_z^2 \Phi + \frac{4\pi N M c^2}{B^2} \left[\frac{d^2 \Phi}{dx^2} + \frac{d\Phi}{dx} \frac{d \ln N}{dx} \right] + \frac{Q(x, \omega)}{r_D^2(x)} \Phi = 0; \quad (11)$$

$$Q(x, \omega) = \kappa T \frac{k_y c}{|e| B} \frac{1}{\omega} \frac{d \ln N}{dx} - 1 + i \sqrt{\frac{\pi}{2}} \frac{1}{|k_z|} \sqrt{\frac{m}{\kappa T}} \left(-\omega + \kappa T \frac{k_y c}{|e| B} \frac{d}{dx} \ln \frac{M}{\sqrt{T}} \right). \quad (12)$$

We note immediately that the imaginary part of the right-hand side of Eq. (12) is assumed to be small

in comparison with the real one. Therefore, in the determination of the frequency, one does not have to consider the imaginary part in the first approximation.

Equation (11) contains two characteristic parameters. Corresponding to this situation, we can distinguish two limiting cases. We take $4\pi N M c^2 \gg B^2$, i.e., we shall assume that the Alfvén velocity is small in comparison with the velocity of light.³⁾ Neglecting the first two terms of Eq. (11), and introducing a new function $\Phi = \psi / \sqrt{N}$, we get

$$\psi'' + \left\{ \frac{Q(x, \omega)}{R^2(x)} - \frac{1}{2} \frac{d^2 N}{dx^2} \frac{1}{N} + \frac{1}{4} \left(\frac{d \ln N}{dx} \right)^2 \right\} \psi = 0. \quad (13)$$

Here $R^2(x) = \kappa T / \Omega_i^2$, where Ω_i is the Larmor frequency of the ions.

Inasmuch as the particle distribution changes little at distances of the order of the Larmor radius, the first component of the curly brackets on the left-hand side of Eq. (13) considerably exceeds the remainder. In just this sense, one can speak of the Larmor radius as a small parameter. Therefore, the asymptotic solution for the potential of the electric field can be written in the form

$$\Phi = (C R^{1/2} / N^{1/2} Q^{1/4}) \exp \left\{ \pm i \int dx \sqrt{Q(x, \omega) / R(x)} \right\}. \quad (14)$$

The corresponding dispersion equation of the plasma oscillations has the form

$$\int dx \sqrt{Q(x, \omega) / R(x)} = \pi n, \quad (15)$$

where the limits of integration are determined as in Eq. (3). Actually, let the dependence $Q(x)$ be such that there is no turning point $Q(x) = 0$ in the plasma. Then the boundary condition of vanishing of the potential on the boundary of the plasma immediately reduces to Eq. (15). The change in the boundary condition leads only to the appearance on the right-hand side of (15) of an additional real component of the order of unity,⁴⁾ which is of little importance. This is connected with the fact that many "wavelengths" of oscillations are included in the characteristic distance for change in the distribution of the plasma.

To make concrete the dependence of the number of particles and the temperature on the coordinates, let us consider the application of the dispersion equation (15). Let $Q(x)/R^2(x) = \text{const}$, which is possible only in the case $T = \text{const}$, $N(x) = N_0 \times$

³⁾The limit $4\pi N M c^2 \ll B^2$ can be considered in a fashion quite similar to what is set forth below.

⁴⁾One can possibly propose special dissipative (or active) boundary conditions leading to an imaginary contribution. We shall not consider such boundary conditions below.

$\exp(-x/x_0)$. We then get the following dispersion equation with the help of (15) for a plasma located in the interval $0 < x < 1$:

$$\frac{\kappa T}{\omega} \frac{k_y c}{|e|B} \frac{d \ln N}{dx} - 1 + i \sqrt{\frac{\pi}{2}} \frac{\omega}{|k_z|} \sqrt{\frac{m}{\kappa T}} \left(\frac{\kappa T}{\omega} \frac{k_y c}{|e|B} \frac{d \ln N}{dx} - 1 \right) = \left(\pi n \frac{R}{L} \right)^2. \quad (16)$$

For the case $n^2 R^2 \ll L^2$ we then get

$$\omega = \kappa T \frac{k_y c}{|e|B} \frac{d \ln N}{dx} = - \frac{\kappa T}{x_0} \frac{k_y c}{|e|B}. \quad (17)$$

Keeping it in mind that the imaginary part of Q is small, we get from Eq. (15) in first approximation:

$$\int \frac{dx}{R(x)} \sqrt{\operatorname{Re} Q(x, \omega)} = \pi n, \quad (18)$$

where the integration is carried out along the real axis, which corresponds to neglect of the small imaginary parts of the coordinates of the turning point.

Taking into account the following term of the expansion in Eq. (15), we get

$$\int \frac{dx}{R(x)} \frac{\operatorname{Im} Q(x, \omega)}{\sqrt{\operatorname{Re} Q(x, \omega)}} = 0. \quad (19)$$

Making use of Eq. (18) as an equation of first approximation, we can materially simplify the analysis of the dispersion equation (15). In order to demonstrate this, we consider the following example. Let $\operatorname{Re} Q(x) = \text{const}$. This can be accomplished only for a number of particles N and temperature T satisfying the equation

$$T d \ln N / dx = -T_0 / x_0 = \text{const}. \quad (20)$$

Then, with the help of Eq. (18), we can easily find ($\omega = \omega' + i\omega''$)

$$\omega' = \frac{k_y c}{|e|B} \kappa T \frac{d \ln N}{dx} \left\{ 1 + \left[\pi n \int \frac{dx}{R(x)} \right]^2 \right\}^{-1}. \quad (21)$$

In the limit $n / \int R^{-1} dx$, small in comparison with unity, the spectrum of (21) is identical with (17). Keeping in mind the smallness of such a parameter and assuming ω' determined by Eq. (21), we can write

$$Q(x) = -i \left[\frac{\omega''}{\omega'} + \sqrt{\frac{\pi}{8}} \frac{\omega'}{|k_z|} \sqrt{\frac{m}{\kappa T}} \frac{d \ln T}{d \ln N} \right] + \left[\pi n \int \frac{dx}{R} \right]^2. \quad (22)$$

If we require the satisfaction of the equation

$$\frac{1}{\sqrt{T}} \frac{d \ln T}{d \ln N} = \frac{C}{\sqrt{T_0}}, \quad (23)$$

then the right side of Eq. (22) is seen to be independent of the coordinates. Then, substituting the

imaginary part of Eq. (22) in Eq. (19), we get

$$\omega'' = - \sqrt{\frac{\pi}{8}} \frac{(\omega')^2}{|k_z|} \sqrt{\frac{m}{\kappa T}} \frac{d \ln T}{d \ln N}. \quad (24)$$

Such a spectrum corresponds to instability of the plasma for $d \ln T / d \ln N < 0$. This conclusion, and also the spectrum of oscillations were obtained by Rudakov and Sagdeev,^[3] who assumed that the dependence of the temperature and number of particles on the coordinate can be arbitrary. The expansion above shows that the spectra obtained by them are possible only in the case of satisfaction of Eqs. (20) and (23), or, what amounts to the same thing, for a particular specific dependence of the temperature and density of the particles on the coordinates.⁵⁾

Satisfaction of Eq. (20) alone is less restrictive. In this case, we again have the spectrum (21) which determines the real part of the frequency of oscillations. For the imaginary part of the frequency, we then get the following equation with the aid of Eqs. (19), (22):

$$\omega'' = - \sqrt{\frac{\pi}{8}} \frac{(\omega')^2}{|k_z|} \left\{ \int \frac{dx}{R(x)} \right\}^{-1} \int \frac{dx}{R(x)} \sqrt{\frac{m}{\kappa T}} \frac{d \ln T}{d \ln N}. \quad (25)$$

Here the condition $d \ln T / d \ln N < 0$, which is satisfied at any point, is by no means sufficient. In fact, in order that the right side of Eq. (25) be greater than zero, which corresponds to the growth of the oscillations in time, the following inequality must hold:

$$\left(\int \frac{dx}{R(x)} \right)^{-1} \int \frac{dx}{R(x)} \frac{1}{\sqrt{T}} \frac{d \ln T}{d \ln N} < 0. \quad (26)$$

Keeping in mind Eq. (20), and also the definition of $R(x)$, we can write this inequality in the form

$$(x_2 - x_1) x_0 \ln [T(x_1)/T(x_2)], \quad (27)$$

where x_1 and x_2 are the boundary points of the plasma (or the turning points).

For definiteness, let $x_2 > x_1$. Then, for $x_0 > 0$, Eq. (27) takes the form $T(x_1) < T(x_2)$. In other words, the temperature increases with increase in x . At the same time, $x_0 > 0$ corresponds to a decrease in density with increase in x . Therefore, the integral jump condition of (27) corresponds not only to the local jump condition of Rudakov and Sagdeev, but also requires that their condition be sufficiently weakly violated throughout the entire plasma.⁶⁾

⁵⁾ $N = N_0 \exp \{ -(2/C) \sqrt{T_0/T} \}$, $\sqrt{T/T_0} = (C/2)(\delta - x/x_0)$; if $C < 0$, then $\delta x_0 < x$, while if $C > 0$, then $\delta x_0 > x$.

⁶⁾ It is necessary to emphasize that all our considerations, which are based on the approximation of the expression for the charge density (10), and which refer to the concrete conformation of the plasma spectrum, are limited to this approximation.

Equation (19) makes it possible to obtain a general expression for the imaginary part of the oscillation frequency,⁷⁾

$$\omega'' = (\omega')^2 \left\{ \int dx \frac{\kappa T(x) k_y c}{R(x) |e| B} \frac{d \ln N}{dx} \frac{1}{\sqrt{\operatorname{Re} Q(x, \omega')}} \right\}^{-1} \times \sqrt{\frac{\pi}{2} \frac{1}{|k_z|}} \sqrt{\frac{m}{\kappa}} \int dx \frac{1}{R(x) \sqrt{\operatorname{Re} Q(x, \omega')}} \frac{1}{\sqrt{T(x)}} \times \left[\frac{k_y c}{|e| B} \kappa T(x) \frac{d}{dx} \ln \frac{N}{\sqrt{T}} - \omega' \right], \quad (28)$$

where, according to Eq. (18), the real part of the frequency is determined by the relation

$$\int \frac{dx}{R(x)} \left[\kappa T(x) \frac{k_y c}{|e| B} \frac{1}{\omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} = \pi n. \quad (29)$$

In the region of transparency considered by us,

$$\kappa T(x) \frac{k_y c}{|e| B} \frac{1}{\omega'} \frac{d \ln N}{dx} \gg 1. \quad (30)$$

Making use of this relation, we can write down the condition for the instability of the plasma, corresponding to the positive nature of the right-hand side of Eq. (28), in the following form:

$$\int \frac{dx}{T(x) \sqrt{\operatorname{Re} Q(x, \omega')}} \times \left[\kappa T(x) \frac{k_y c}{|e| B} \frac{1}{\omega'} \left(\frac{d \ln N}{dx} - \frac{1}{2} \frac{d \ln T}{dx} \right) - 1 \right] > 0. \quad (31)$$

Here and below, the integration is carried out from small values to large ones.

In the special case $T = \text{const}$, the relation (31) takes the form

$$\frac{1}{R} \int dx \left[\frac{\kappa T k_y c}{|e| B \omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} > 0. \quad (32)$$

In accord with the relation (30), the inequality (32) is violated only in one isolated case when Eq. (20) is satisfied (and $n = 0$). Thus one can establish the fact that a weakly inhomogeneous plasma with a constant temperature $T_e \gg T_i$, confined by a strong magnetic field of plane geometry, can be unstable. The frequency of the increasing oscillations is determined by the equation

$$\int dx \left[\frac{\kappa T k_y c}{|e| B \omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} = \pi n R. \quad (33)$$

We note that as is seen from the inequality (31), the requirement of a decrease of the temperature of the plasma in the same direction in which the density decreases in the example under consideration corresponds to stabilization of the plasma but

cannot be used as a sufficient condition for stability.

Finally, we write down the dispersion equation for an arbitrary distribution function for the electrons of the form $f_0 = N(x) F(\mathcal{E}, x)$, where $\mathcal{E} = mv_z^2/2$:

$$\frac{|e| B}{V M c} \int dx \left[\frac{k_y c}{B |e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{1/2} = \pi n, \quad (34)$$

$$\omega'' = \pi \frac{(\omega')^2}{|k_z|} \left\{ \int dx \frac{d \ln N}{dx} \left[\frac{k_y c}{B |e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} \right\}^{-1} \times \int dx \left[\frac{k_y c}{B |e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} \times \left[\omega' \frac{B |e|}{k_y c} \frac{\partial F(0, x)}{\partial \mathcal{E}} + \frac{d \ln N}{dx} F + \frac{\partial F(0, x)}{\partial x} \right]. \quad (35)$$

Here $\langle 1/mv^2 \rangle = - \int_{-\infty}^{+\infty} dv_z \partial F / \partial \mathcal{E}$. If this quantity is positive, then the condition for instability can be written in the following form:

$$\int dx \left(\frac{\partial F(0, x)}{\partial \mathcal{E}} + \left[\frac{d \ln N}{dx} F + \frac{\partial F(0, x)}{\partial x} \right] \frac{k_y c}{B |e|} \frac{1}{\omega'} \right) \times \left[\frac{k_y c}{B |e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} > 0. \quad (36)$$

The local condition for instability can be written by requiring the positiveness of the integrand

$$\frac{\partial F(0, x)}{\partial \mathcal{E}} + \left[\frac{d \ln N}{dx} F + \frac{\partial F(0, x)}{\partial x} \right] \frac{k_y c}{B |e|} \frac{1}{\omega'} > 0. \quad (37)$$

Such a local condition is necessary but by no means sufficient. Instability will of course take place if one requires the fulfilment of the condition (37) throughout the whole region of transparency. But such a requirement is rather excessive, as is evident from the relation (36).

Comparing the results of this section with the results of the work of Rudakov and Sagdeev,^[3] it should be noted that in our consideration the frequency of the oscillations is a number determined only by the integral parameters of the plasma. On the other hand, the frequency in ^[3] is a point function.⁸⁾

4. LOW FREQUENCY OSCILLATIONS IN A CYLINDRICAL PLASMA COLUMN CONFINED BY A MAGNETIC FIELD

This section will also be devoted to a plasma confined by a magnetic field (with $T_e \gg T_i$). How-

⁸⁾M. L. Levin has pointed out the well-known mechanical analog which characterizes the state of affairs. If we have a set of pendulums on different suspension threads, then each of them possesses a characteristic frequency (point function). The presence of even small coupling leads, generally speaking, to new characteristic frequencies which are no longer point functions and which are determined by the parameters of the set of pendulums as a whole. I take this occasion to express my gratitude to M. L. Levin for this useful observation.

⁷⁾In the case $\omega'' \ll \omega'$, Eq. (19) takes the form

$$\int \frac{dx}{R} \frac{1}{\sqrt{\operatorname{Re} Q}} \left\{ \operatorname{Im} Q(x, \omega') + \omega'' \frac{\partial}{\partial \omega'} \operatorname{Re} Q(x, \omega') \right\} = 0.$$

ever, in contrast to the previous section, we shall consider here a cylindrical plasma column confined by a magnetic field which depends only on the radial coordinate. It is assumed that the particle distribution in the ground state of the plasma has the form

$$f_0(v_z, v_\perp, r) = (v_\perp/\Omega) \sin(\psi - \varphi) \partial f_0/\partial r, \quad (38)$$

where $\mathbf{r} = (r, \psi, z)$, $\mathbf{v} = (v_\perp, \varphi, v_z)$ and the z axis is directed along the direction of the magnetic field.

We assume that the plasma pressure (more precisely, the transverse pressure)

$$P = \sum \int d\mathbf{v} \frac{mv_\perp^2}{2} f_0 \quad (39)$$

significantly exceeds the magnetic pressure. Then, using the relation

$$\frac{d \ln B^2}{dr} + \frac{P}{B^2/8\pi} \frac{d \ln P}{dr} = 0, \quad (40)$$

we can, as in the previous section, neglect the dependence of the magnetic field on the coordinate in comparison with the sensitive dependence of the distribution on consideration of the motion of the particles of the plasma.

It is not difficult to establish the fact that, within the framework of the assumptions used in obtaining the formula (10), the following expression results for the charge density of the ions in the case of cylindrical symmetry:

$$\frac{e^2 N}{M\Omega^2} \left[\frac{d^2 \Phi}{dr^2} + \frac{1}{r} \frac{d\Phi}{dr} \left(1 + \frac{d \ln N}{d \ln r} \right) - \frac{l^2}{r^2} \Phi \right] - \Phi \frac{e^2}{M} \int_{-\infty}^{+\infty} \frac{dv_z}{\omega - k_z v_z} \left\{ k_z \frac{\partial \tilde{f}_0}{\partial v_z} - \frac{l}{\Omega} \frac{1}{r} \frac{\partial \tilde{f}_0}{\partial r} \right\}. \quad (41)$$

Here Φ depends only on r . The dependence on the time and the other coordinates taken in the form $\exp(-i\omega t + ik_z z + i\psi)$, is separated out and will not be considered further.

Considering that $4\pi N m c^2 \gg B^2$, and also limiting ourselves to the case of the projection of the phase velocity on the z axis that is small in comparison with the thermal velocity of the electrons and large in comparison with the sound velocity, we get the following equation for the field potential Φ ;

$$\frac{d^2 \Phi}{dr^2} + \frac{1}{r} \frac{d\Phi}{dr} \left(1 + \frac{d \ln N}{d \ln r} \right) - \frac{l^2}{r^2} \Phi + \frac{Q(r, \omega)}{R^2(r)} \Phi = 0, \quad (42)$$

where for a Maxwellian electron distribution

$$Q(r, \omega) = \kappa T(r) \frac{lc}{B|e|} \frac{1}{\omega} \frac{1}{r} \frac{d \ln N}{dr} - 1 + i \sqrt{\frac{\pi}{2}} \frac{\omega}{|k_z|} \times \sqrt{\frac{m}{\kappa T(r)}} \left[\kappa T(r) \frac{lc}{B|e|} \frac{1}{\omega} \frac{1}{r} \frac{d \ln N}{dr} \ln \frac{N}{\sqrt{T}} - 1 \right]. \quad (43)$$

Making the substitution $\psi = \Phi \sqrt{rN}$, we get

$$\psi'' + \left\{ \frac{Q(r, \omega)}{R^2(r)} - \frac{l^2}{r^2} + \frac{1}{4} \left(\frac{1}{r} - \frac{N'}{N} \right)^2 - \frac{1}{2} \frac{N''}{N} \right\} \psi = 0. \quad (44)$$

We shall assume lR to be small.⁹⁾ Then the asymptotic solution for the potential of the electric field has the form

$$\Phi = \frac{CR^{1/2}}{r^{1/2} N^{1/2} Q^{1/4}} \exp \left\{ \pm i \int dr \frac{\sqrt{Q(r, \omega)}}{R(r)} \right\}, \quad (45)$$

while the corresponding dispersion equation which defines the oscillation spectrum of the plasma can be written in the form

$$\int \frac{dr}{R(r)} \sqrt{Q(r, \omega)} = \pi n. \quad (46)$$

Inasmuch as the imaginary part of $Q(r, \omega)$ is small, one can write the following equation defining the real part of the oscillation frequency:

$$\int \frac{dr}{R(r)} \left[\frac{\kappa T}{\omega'} \frac{lc}{B|e|} \frac{1}{r} \frac{d \ln N}{dr} - 1 \right]^{1/2} = \pi n. \quad (47)$$

For the imaginary part of the frequency of oscillation, we get the following equation directly:

$$\begin{aligned} \omega'' &= \sqrt{\frac{\pi}{2}} \frac{(\omega')^3}{|k_z|} \sqrt{\frac{m}{\kappa}} \\ &\times \left\{ \int \frac{dr}{R(r)} \frac{1}{\sqrt{\operatorname{Re} Q(r, \omega)}} \kappa T(r) \frac{lc}{B|e|} \frac{1}{r} \frac{d \ln N}{dr} \right\}^{-1} \\ &\times \int \frac{dr}{R(r)} \frac{1}{\sqrt{\operatorname{Re} Q(r, \omega)}} \frac{1}{\sqrt{T(r)}} \\ &\times \left[\frac{\kappa T(r)}{\omega'} \frac{lc}{B|e|} \frac{1}{r} \frac{d \ln N}{dr} \ln \frac{N}{\sqrt{T}} - 1 \right]. \end{aligned} \quad (48)$$

It is evident that the following inequality holds in the region of transparency over which the integration here is carried out:

$$\frac{\kappa T}{\omega'} \frac{lc}{B|e|} \frac{1}{r} \frac{d \ln N}{dr} \geq 1. \quad (49)$$

Making use of this relation, we obtain the following condition for positive character of the imaginary part of the frequency (the instability condition):

$$\int \frac{dr}{T(r)} \frac{1}{\sqrt{\operatorname{Re} Q(r, \omega)}} \left(\frac{\kappa T(r)}{\omega'} \frac{lc}{B|e|} \frac{1}{r} \frac{d \ln N}{dr} \ln \frac{N}{\sqrt{T}} - 1 \right) > 0. \quad (50)$$

In a fashion similar to the formula (3.2), the integration here and below is carried out in the direction from small r to large.

In the case of a plasma temperature independent of the coordinates the inequality (50) has the form

$$\int dr \left[\frac{\kappa T}{\omega'} \frac{lc}{|e|B} \frac{1}{r} \frac{d \ln N}{dr} - 1 \right]^{1/2} = \pi |n| R > 0, \quad (51)$$

which is always satisfied (except in the case $N \sim e^{-r^2/r_0^2}$ and the value $n = 0$ is possible here).

⁹⁾In the case of large lR/r in the formulas written out below we must have $Q(r, \omega) = (lR/r)^2$ in place of $Q(r, \omega)$.

Consequently, a plasma column with an electron distribution function of the form (38) and which has a temperature that is homogeneous over a cross section confined by the cylindrical magnetic field with straight lines of force is seen to be unstable relative to the oscillations (51).

In conclusion, let us write the dispersion equations for a non-Maxwellian electron distribution function of the form $\tilde{f}_0 = N(r) F(\mathcal{E}, r)$:

$$\frac{|e|B}{\sqrt{Mc}} \int dr \left[\frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \frac{d \ln N}{dr} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{1/2} = \pi n, \quad (52)$$

$$\begin{aligned} \omega'' = \pi \frac{(\omega')^2}{|k_z|} \left\{ \int \frac{dr}{r} \frac{d \ln N}{dr} \left[\frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \frac{d \ln N}{dr} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} \right\}^{-1} \\ \times \int dr \left\{ \omega' \frac{B|e|}{lc} \frac{\partial F(0, r)}{\partial \mathcal{E}} + \frac{1}{r} \left[\frac{d \ln N}{dr} F + \frac{\partial F(0, r)}{\partial r} \right] \right\} \\ \times \left[\frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \frac{d \ln N}{dr} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2}. \end{aligned} \quad (53)$$

If it is taken into account that $\langle 1/mv^2 \rangle$ is positive, then one can proceed further and write down the following stability condition:

$$\begin{aligned} \int dr \left\{ \frac{\partial F(0, r)}{\partial \mathcal{E}} + \frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \left[\frac{d \ln N}{dr} F + \frac{\partial F(0, r)}{\partial r} \right] \right\} \\ \times \left[\frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \frac{d \ln N}{dr} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2}. \end{aligned} \quad (54)$$

By requiring that the integrand be less than zero

$$\frac{\partial F(0, r)}{\partial \mathcal{E}} + \frac{lc}{B|e|} \frac{1}{\omega'} \frac{1}{r} \left[\frac{d \ln N}{dr} F + \frac{\partial F(0, r)}{\partial r} \right] < 0, \quad (55)$$

we obtain the local stability condition. This condition is necessary in contrast with the sufficient condition (54). If we require the satisfaction of the condition (55) throughout the region of transparency (in other words, to consider it as non-local), then the stability will take place in general earlier than such a condition is seen to be satisfied.

Here the problem does not arise as to what measure it is exhaustive to consider the problem of the stability of the plasma. However, one of the reasons for writing this paper were the prospects uncovered by the possibility of applying the method of "quasi-classical quantization" to the problem of plasma stability.

5. CONCLUSION. GENERALIZATION

Summing up the situation briefly, one can repeat and generalize several premises of our research.

The first from which we set out, is that the eigenvalue problem which arises in the question of interest to us, concerning the natural oscillations of weakly inhomogeneous plasma, is solved by means of the "quantization rules" which are similar to the quasi-classical quantization rules of

Bohr. It must be noted that it is not difficult to ascertain from what has been set forth above that the formula (3) can be written in the form

$$\oint dx k_x(\omega, x) = 2\pi n, \quad (56)$$

where $k_x(\omega, x)$ is the projection of the wave vector on the x axis and is of course a function of the coordinates.

Keeping in mind the deep analogy between mechanics and geometric optics,^[4] one can establish the fact that for finite, conditionally periodic trajectories of rays in an inhomogeneous medium, the eikonal¹⁰⁾ is a non-unique quantity defined with accuracy up to sums of multiples of the values of the analogs of the mechanical action variables (the eikonal variables). Therefore, the problem of obtaining the spectrum of the characteristic oscillations in a non-one-dimensionally weakly inhomogeneous plasma reduces to finding the conditional periodic motions of the rays and, what is very important, the adiabatic invariance of the eikonal variables I_α corresponding to them. The consequent quantization rule is that the eikonal variables can take on only values which are multiples of 2π :

$$I_\alpha = 2\pi n_\alpha. \quad (57)$$

In this paper we have considered only the application of a one-dimensional "quantization rule" (56). The characteristic feature of the natural oscillations considered is the smallness of their "wavelength" in comparison with the distances over which the particle distribution of an equilibrium plasma changes appreciably. On the other hand, the "wavelength" is large in comparison with such "microscopic" scales of a weakly inhomogeneous plasma as the Debye or Larmor radii. The latter fact appears in the result that the "quasi-classical quantum number" is large in comparison with unity.

In the examples considered above we have limited ourselves to the case of a medium without absorption and a medium with weak absorption (or weak buildup). For the case of strong absorption or strong buildup the projection of the wave vector in formula (56) can possess a large imaginary part, while the turning points appear to be located in the complex plane. In such a case, the difference between the regions of transparency and opacity disappear. However, even in this case, by following

¹⁰⁾In geometric optics, the field has the form $\Phi = Ae^{i\psi}$, where A is a slowly changing function and ψ is the eikonal — a very large quantity which appears in analogy to the action function divided by Planck's constant.

[2], we can make clear the conditions under which the quantization rule (56) becomes possible.

In conclusion, I express my thanks with great pleasure to A. A. Rukhadze, E. E. Lovetskiĭ, and L. M. Kovrizhnykh, useful conversations with whom promoted my interest in the problems of the theory of the stability of a magnetically confined plasma.

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STATIONARY PLASMA FLOW IN A MAGNETIC FIELD

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Isolated waves of finite amplitude propagating in a cold plasma at an arbitrary angle with respect to the magnetic field are analyzed. It is shown that three waves of this kind are possible: large scale compressional shocks (Alfvén waves), rarefaction shocks (magnetic sound waves), and small-scale compressional shocks corresponding to high-frequency magnetic-sound waves. The relation between the wave amplitude and the Mach number is established. The critical magnetic field at which the solution for the compressional wave becomes unstable is also determined.

1. INTRODUCTION

MEDIA in which the dispersion is nonlinear for small oscillations can support the propagation of stationary isolated waves: if the phase velocity of the waves diminishes (increases) with increasing wave number, the stationary isolated wave is a density compression (rarefaction).^[1]

In the present work we consider an isolated wave propagating in a cold low-density plasma (so that dissipation processes can be neglected) at an arbitrary angle with respect to the unperturbed magnetic field H_0 . The case being treated is of interest as it exhibits several oscillation branches. In the low frequency region ($\omega \lesssim \Omega_i = eH_0/m_i c$, Ω_i is the ion Larmor frequency) these are accelerated waves ("magnetic" sound) with a minimum phase velocity $v_+ = H_0/\sqrt{4\pi n_0 m_i}$, and retarded Alfvén waves with a maximum phase velocity $v_- = v_+ \cos \theta$ (here θ is the angle between the direction of propagation and the magnetic field).

The qualitative behavior of ω/k as a function of frequency is shown in Fig. 1 ($\cos \theta \gg \sqrt{m_e/m_i}$, cf. [2]). The existence of two branches means that the isolated wave can be a compression or rarefaction shock.

In the high-frequency region ($\Omega_i \ll \omega \ll \Omega_e$) there is one branch with a high phase velocity $\omega/k = v_+ \sqrt{\omega/\Omega_i}$. The characteristic frequency here is $\sqrt{\Omega_e \Omega_i}$ while the effective value of the magnetic field in the wave is of order $H_0 \sqrt{m_i/m_e}$ so that $\omega/k \sim v_+ \sqrt{m_i/m_e}$. Higher propagation velocities mean that the change in the various physical quantities in the wave can be anomalously large. The theorem given in [1] does not hold here and if the fine structure of the wave at its base is neglected the wave is always a compression shock.

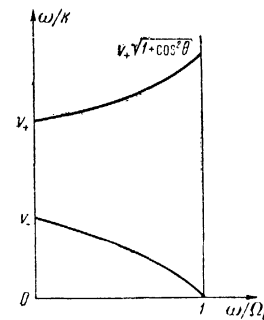


FIG. 1

The limiting cases indicated above differ appreciably in their propagation scales. The flow of plasma must be symmetric with respect to the peak of the wave in the coordinate system moving with the wave since there is no dissipation. It is clear that in the case of Alfvén and magnetic-sound shocks (large-scale waves) the ions cannot execute very many rotations in the magnetic field of the shock without disturbing the correlation between the trajectories of the heavy particles at the "input" and "output" of the wave. Taking the ion Larmor radius as a characteristic dimension of the wave we find

$$v_+ / \Omega_i = \sqrt{m_i c^2 / 4\pi e^2 n_0}.$$

The wavelength of the high-frequency shock (small-scale wave) is determined by the electron Larmor radius and is of order $\sqrt{m_e c^2 / 4\pi e^2 n_0}$.

In this work we consider the structure of the isolated wave indicated above and establish the relation between wave amplitude and Mach number. We also find the limiting values for various physical quantities and the limitations on the Mach number. We note that the problem of the isolated wave has been considered earlier for the case $\theta = \pi/2$

in ^[1,3,4] and for the case $\theta = 0$ in ^[5-7]. Finally, a low-amplitude rarefaction shock has been investigated by Galeev and Karpman.^[8]

2. EQUATIONS OF MOTION AND CONSERVATION LAWS

In the present work we limit ourselves to the nonrelativistic case: $v_0/c \ll 1$, where v_0 is the propagation velocity of the wave. Then, the initial system of equations of motion for the electrons and ions and Maxwell's equations are

$$\frac{\partial \mathbf{v}_{e,i}}{\partial t} + (\mathbf{v}_{e,i} \nabla) \mathbf{v}_{e,i} = \mp \frac{e}{m_{e,i}} \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_{e,i} \mathbf{H}] \right), \quad (1)^*$$

$$\partial n_{e,i} / \partial t + \operatorname{div} n_{e,i} \mathbf{v}_{e,i} = 0, \quad (2)$$

$$\operatorname{rot} \mathbf{H} = \frac{4\pi e}{c} (n_i \mathbf{v}_i - n_e \mathbf{v}_e), \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad (3)^\dagger$$

$$\operatorname{div} \mathbf{E} = 4\pi e (n_i - n_e), \quad \operatorname{div} \mathbf{H} = 0. \quad (4)$$

We assume further that all quantities depend only on $\xi = z + v_0 t$, with $\mathbf{H} \rightarrow \mathbf{H}_0$, $\mathbf{v}_{e,i} \rightarrow 0$, and $n_{e,i} \rightarrow n_0$ as $\xi \rightarrow \pm \infty$. The system in (1)–(4) can now be written conveniently in complex form:

$$\rho_{e,i} \frac{dv_{e,i}}{d\xi} = \mp \frac{ie}{m_{e,i} c} (H \rho_{e,i} - H_\perp^0 - H_\parallel^0 v_{e,i} / v_0), \quad (5)$$

$$\rho_{e,i} \frac{dv_{e,i}^\perp}{d\xi} = \mp \frac{e}{m_{e,i} v_0} \left[E_\parallel + \frac{1}{c} \operatorname{Im} (v_{e,i}^* H) \right], \quad (6)$$

$$idH / d\xi = 4\pi n_0 c^{-1} (v_i / \rho_i - v_e / \rho_e), \quad (7)$$

$$dE_\parallel / d\xi = 4\pi n_0 (1/\rho_e - 1/\rho_i),$$

$$\rho_{e,i} \equiv n_0 / n_{e,i} = 1 + v_{e,i}^\perp / v_0; \quad \mathbf{v}_{e,i} \equiv (\operatorname{Re} v_{e,i}, \operatorname{Im} v_{e,i}, v_{e,i}^\perp),$$

$$\mathbf{H} \equiv (\operatorname{Re} H, \operatorname{Im} H, H_\parallel^0). \quad (8)$$

The projections of various vectors on the z axis are denoted by $\parallel$ while the projections on the x axis are denoted by $\perp$.

The equations in (5)–(8) have the following first integrals (conservation of momentum):

$$m_e v_e + m_i v_i = H_\parallel^0 (H - H_\perp^0) / 4\pi n_0 v_0, \quad (9)$$

$$m_e v_e + m_i v_i = \{E_\parallel^2 + |H_\perp^0|^2 - |H|^2\} / 8\pi n_0 v_0. \quad (10)$$

We shall treat in detail the quasi-neutral case:

$$\rho_e \approx \rho_i = \rho, \quad v_e^\perp \approx v_i^\perp \approx v_\perp, \quad E_\parallel \approx 0.$$

The quasi-neutrality of a plasma is guaranteed over a wide range of angles θ by the fact that the problem is nonrelativistic. It follows from (10) that in this case

$$\rho = 1 + (|H_\perp^0|^2 - |H|^2) / 8\pi n_0 m_i v_0^2. \quad (11)$$

* $[\mathbf{vH}] = \mathbf{v} \times \mathbf{H}$.

$\dagger \operatorname{rot} = \operatorname{curl}$.

3. LARGE-SCALE WAVES

The drift approximation can be used for the description of electron motion in large-scale waves. The only limitation is on the range of variation of θ : the condition $\cos \theta \gg \sqrt{m_e/m_i}$ must hold.

From (5) and (10) we find that the electron and ion velocities and the magnetic fields are related by

$$v_e = v_0 (H \rho - H_\perp^0) / H_\parallel^0, \quad (12)$$

$$v_i = \mu_\parallel^2 v_0 (H - H_\perp^0) / H_\parallel^0, \quad (13)$$

where $\mu = v_+/v_0$ is the reciprocal Mach number ($\mu_\parallel, \mu_\perp \equiv \mu \cos \theta, \mu \sin \theta$).

It will now be convenient to write all quantities in dimensionless form. We introduce the notation:

$$H = H_\perp^0 \sqrt{\lambda} e^{i\varphi}, \quad |H / H_\perp^0| = \sqrt{\lambda}, \quad \xi = \xi_i \tau,$$

$$\xi_i = \mu_\parallel \sqrt{m_i c^2 / 4\pi e^2 n_0}.$$

Equation (7) now becomes

$$i\rho(\lambda) \frac{d}{d\tau} (\sqrt{\lambda} e^{i\varphi}) + \sqrt{\lambda} e^{i\varphi} (\rho(\lambda) - \mu_\parallel^2) + \mu_\perp^2 - 1 = 0. \quad (14)$$

Here $\rho(\lambda) = 1 + \frac{1}{2} \mu_\perp^2 (1 - \lambda)$. Multiplying Eq. (14) by $d(\sqrt{\lambda} e^{-i\varphi})$ and taking the real part, we have after integrating using the boundary conditions $\lambda(\pm \infty) = 1$ and $\varphi(\pm \infty) = 0$

$$(1 - \mu_\parallel^2) (\lambda - 2\sqrt{\lambda} \cos \varphi + 1) = \mu_\perp^2 (\lambda - 1)^2 / 4. \quad (15)$$

Using the last expression we can eliminate the function of φ in (14). Thus we obtain the following equation for λ :¹⁾

$$\rho(\lambda) d\lambda / d\tau = \pm \frac{1}{4} \mu_\perp^2 |\lambda - 1| \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)},$$

$$\lambda_\pm = 1 + 4\mu_\perp^2 (1 - \mu_\parallel^2 \pm \mu_\perp \sqrt{1 - \mu_\parallel^2}). \quad (16)$$

Here, $\sqrt{\lambda_+}$ is the maximum value of the magnetic field in the compressional wave, $\sqrt{\lambda_-}$ is the minimum magnetic field in the rarefaction wave. It is evident from Eq. (16) that the isolated wave can be either a compression shock or a rarefaction shock. Since λ is real and $\lambda \rightarrow 1$ when $\tau \rightarrow \pm \infty$, we must satisfy the condition $\operatorname{Im} \lambda_\pm = 0$ and $\lambda_\pm \geq 1$, which can be rewritten:

$$v_- < v_0 < v_+. \quad (17)$$

One of the conditions in (17) is obvious and the other means that the velocity of the compression (rarefaction) wave must be smaller (larger) than the phase velocity of the accelerated (retarded) waves.

The qualitative behavior of $\sqrt{\lambda_\pm(\mu)}$ is shown

¹⁾In Eq. (16) we must use both signs since we require a symmetric solution and $d\lambda/d\tau$ has different signs with respect to the center of symmetry $\tau = 0$.

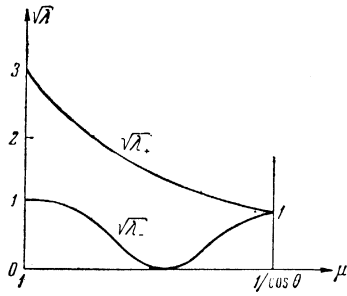


FIG. 2

in Fig. 2. It is evident that the relation between amplitude and Mach number for the ($\rho > 1$) is not unique.

In the case of the rarefaction waves (diamagnetic wave or $\lambda < 1$) with $\mu = 2/\sqrt{1 + 3 \cos^2 \theta}$ the amplitude of the transverse magnetic field vanishes at the extremum while the density is given by the expression

$$(n/n_0)_{min} = (1 + 3 \cos^2 \theta) / (3 + \cos^2 \theta). \quad (18)$$

In the paramagnetic wave ($\lambda > 1$) the amplification of the transverse magnetic field can reach a maximum of a factor of three. In this case $\mu = 1$ while the density is given by

$$(n/n_0)_{max} = (1 - 4 \sin^2 \theta)^{-1}. \quad (19)$$

The density vanishes at $\theta = 30^\circ$. At this point the solution no longer holds. This situation arises because we have neglected the thermal spread in the velocities of the plasma particles. Actually, the plasma flux at the compression point is retarded and transfers its energy to the magnetic field. If the retardation is strong (the plasma essentially comes to rest $v_{||} + v_0 \approx 0$), because of the thermal spread there is flux of reflected particles and this is obviously not taken into account by the equations of motion (1). The criterion for the single-velocity approximation can be written as follows:

$$H_0^2 \gg 8\pi n_0 T \rho^{-\gamma}. \quad (20)$$

Here T and γ are the temperature and adiabatic index of the plasma.

Thus, when $\theta \geq 30^\circ$ the maximum magnetic field in the paramagnetic wave (and this also applies to the wave velocity v_0) must be rather small. From the condition $\rho > 0$ we find that inequality (17) and $1 < \lambda_+ < 9$ for the compression wave must be replaced by

$$v_- < v_0 < v_+ \sqrt{\frac{2 \cos^2 \theta}{1 + \sin \theta}}, \quad 1 < \lambda_+ < 1 + \frac{4 \operatorname{ctg}^2 \theta}{1 + \sin \theta}, \quad \theta \geq 30^\circ. \quad (21)^*$$

* $\operatorname{ctg} = \cot$.

The upper limits on v_0 and λ_+ in the last inequalities are determined taking account of (20).

Equation (16) can be integrated in terms of elementary functions. Since ($\lambda > 1$) we have for the compression wave

$$\frac{\operatorname{arccosh} [1 + 2(1 - \lambda_-)(\lambda_+ - \lambda) / (\lambda - 1)(\lambda_+ - \lambda_-)]}{2 \sqrt{(\mu^2 - 1)(1 - \mu_{||}^2)}} - \operatorname{arccos} \left(1 + 2 \frac{\lambda - \lambda_+}{\lambda_+ - \lambda_-} \right) = \tau. \quad (22)^*$$

We now write expressions for the physical quantities in dimensional form. The following formulas hold for a low amplitude compression wave ($1 - \mu_{||}^2 = \frac{1}{4} \delta^2 \tan^2 \theta$):

$$n = n_0 \left(1 + \frac{\delta \operatorname{tg}^2 \theta}{\operatorname{ch} \alpha} \right), \quad v_{||} = -v_0 \frac{\delta \operatorname{tg}^2 \theta}{\operatorname{ch} \alpha}, \quad v_0 = v_-, \quad (23)^\dagger$$

$$v_e \approx v_i = \sin \theta v_+ F(\xi), \quad (24)$$

$$H = H_0^0 (1 + \delta / \operatorname{ch} \alpha) (1 + F(\xi)),$$

$$F(\xi) = \frac{1}{\operatorname{ch}^2 \alpha} (|\operatorname{sh}^2 \alpha - 1| + 2i \operatorname{sh} \alpha) - 1, \quad \alpha = \delta \operatorname{tg}^2 \theta \cdot \xi / \xi_i. \quad (25)^\ddagger$$

The solution for the rarefaction wave is obtained from (22) by replacing $\lambda_{\pm}$ by $\lambda_{\mp}$ and $\lambda - 1$ by $1 - \lambda$.

We now find the quasi-neutrality condition for the plasma. It follows from Eq. (6) that

$$E_{||} \sim v_0 H_0 / c \cos \theta. \quad (26)$$

Substituting $E_{||}$ in Eq. (8) we find that the quasi-neutrality condition for the plasma is

$$|\rho_e - \rho_i| \sim (v_0 / c \cos \theta)^2 \ll 1. \quad (27)$$

4. SMALL-SCALE WAVES

We now consider small-scale waves. We assume at the outset that the parameter $\epsilon = \sqrt{m_e / m_i} \cos^2 \theta$ is small. In this case the drift approximation cannot be used to describe the electron motion since the wave length is of the order of the electron Larmor radius. However, the calculations can be simplified considerably if we limit ourselves to the analysis of waves whose amplitude is not too small.

Since the electrons execute about one orbit in the magnetic field of the wave for the conditions that hold here, the electron angular momentum cannot be smaller than the ion angular momentum: $m_e v_e \sim m_i v_i$.

For this reason we can neglect the ion current in Maxwell's equations. The characteristic value of the induced magnetic field in the wave is H

* $\operatorname{ch} = \cosh$.

† $\operatorname{tg} = \tan$.

‡ $\operatorname{sh} = \sinh$.

$\sim H_0/\epsilon$. Hence, to accuracy of order ϵ we can neglect H_1^0 everywhere compared with H and take the condition $H(\pm\infty) = 0$ as the boundary condition on H . Thus, H satisfies the following equation approximately

$$\rho \frac{d}{d\xi} \left(\rho \frac{dH}{d\xi} \right) + \frac{ieH_1^0}{m_e c v_0} \rho \frac{dH}{d\xi} - \frac{4\pi e^2 n_0}{m_e c^2} \rho H = 0, \quad (28)$$

which is equivalent to the equation of motion of a nonlinear oscillator in a magnetic field.

From momentum and energy conservation we have

$$\left| \rho \frac{dH}{d\xi} \right|^2 = \frac{4\pi e^2 n_0}{m_e c^2} |H|^2 \left(1 - \frac{|H|^2}{16\pi n_0 m_e v_0^2} \right), \quad \rho \frac{d \arg H}{d\xi} = \frac{eH_1^0}{2m_e c v_0}. \quad (29)$$

Turning to dimensionless variables we write

$$|H| = H_0 v / \epsilon M; \quad M = H_1^0 / \sqrt{16\pi n_0 m_e v_0^2}, \\ \xi = \xi_e \eta, \quad \xi_e = \sqrt{m_e c^2 / 4\pi e^2 n_0}. \quad (30)$$

From Eq. (29) we obtain the following equation for ν

$$\rho(\nu) d\nu / d\eta = \pm \nu \sqrt{\nu_+^2 - \nu^2}, \quad (31)$$

where $\nu_+ = \sqrt{1 - M^2}$ is the maximum value of the magnetic field in the wave.

In the approximation used here we have $\rho(\nu) = 1 - 2\nu^2$. From the conditions $\rho > 0$ and $\text{Im } \nu_+ = 0$ we obtain the following limitations on the reciprocal Mach number:

$$1/\sqrt{2} < M < 1. \quad (32)$$

When ν_+ is approximately $1/\sqrt{2}$ the particle density at the maximum again becomes large. The condition that must be satisfied if one is to neglect the thermal spread of the velocities in this case is less stringent than in Eq. (20):

$$H_0^2 \gg 8\pi n_0 e^2 T \rho^{-\gamma}. \quad (33)$$

Integrating Eq. (31) we have

$$\text{arcch } \nu_+ / \nu - 2\nu_+ \sqrt{\nu_+^2 - \nu^2} = \nu_+ |\eta|. \quad (34)$$

For amplitudes that satisfy the condition $\epsilon \ll \nu_+ \ll 1$ the second term in the left side of Eq. (34) can be neglected. Returning to the physical variables, we obtain the following expressions:

$$v_{\parallel} = -2v_0 |f|^2, \quad n = n_0 (1 + 2|f|^2), \quad v_0 = H_1^0 / \sqrt{16\pi n_0 m_e}, \\ H = \sqrt{m_i / m_e} H_1^0 f(\xi), \quad v_e = 2 \sqrt{m_i / m_e} v_0 f(\xi), \\ v_i \sim \sqrt{m_e / m_i} v_0 f(\xi), \quad (35)$$

where $f(\xi) = \nu_+ e^{i\xi/\xi_e} / \coth(\nu_+ \xi / \xi_e)$. Here, we have neglected the difference in the velocity of rotation of the phase of the magnetic field and the electrons, a procedure that is valid only for low-amplitude waves. It is evident from Eq. (35) that the energy in the wave is distributed equally between the magnetic field and the translational and rotational motion of the plasma. Since the electron velocity is assumed to be nonrelativistic, the following condition must be satisfied:

$$H_0 \ll \sqrt{4\pi n_0 m_e^2 c^2 / m_i}. \quad (36)$$

Estimates analogous to those in Eqs. (26) and (27) show that the quasi-neutrality of the plasma is also provided by the nonrelativistic condition for the problem (36).

The author is indebted to V. L. Pokrovskii for a number of valuable remarks.

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EXCITATION OF ION-ACOUSTIC WAVES AND ELECTRON HEATING IN A PLASMA IN AN EXTERNAL ELECTRIC FIELD

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We consider a plasma with cold ions in a strong external electric field that is periodic in time. The collisionless mechanism responsible for electron heating under these conditions is analyzed. This heating mechanism operates as follows: because of the external electric field the electrons acquire a velocity with respect to the ions; ion-acoustic waves are excited and these tend to smooth the electron distribution function. The smoothing process is repeated in the periodic electric field, leading to flattening of the electron distribution function (consequently an increase in electron temperature) in the velocity range corresponding to twice the velocity acquired by the electrons in the external field.

VARIOUS kinds of waves can be excited in a plasma when the static particle distribution function departs from equilibrium. If, for example, a strong current flows through the plasma, i.e., the ions and electrons (with Maxwellian velocity distributions) move with respect to each other with a constant velocity greater than some critical value, one finds that ion-acoustic waves are excited.^[1] The quasi-linear theory^[2,3] takes account of the feedback effect of the growing waves on the particle distribution function for the weakly unstable case, where the growth rate is small compared with the frequency and the energy density of the wave is small compared with the thermal energy density of the particles. Under these conditions only the fine details of the particle distribution function are affected (for example, when ion-acoustic waves are excited the electron distribution function is flattened in a narrow velocity region about the velocity of the ion-acoustic wave). The gross plasma parameters, such as the moments of the distribution function, are not affected significantly. Below we describe a case in which the instability is weak but in which one of the gross parameters, i.e., the electron temperature, is changed significantly.

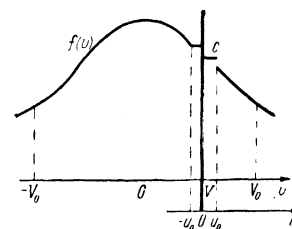
We shall treat a spatially uniform problem: a plasma in a strong magnetic field in which the ions and electrons can be described by a one-dimensional distribution function $f(v)$, where v is the velocity in the direction of the magnetic field.

Let us assume that there is an external electric field $E = E_0 \cos \omega t$ in the same direction as the magnetic field. Under the effect of this field the electrons move with respect to the ions and if $T_i \ll T$ (T_i , T are respectively the ion and elec-

tron temperatures) ion-acoustic waves are excited in the plasma; these waves are characterized by a phase velocity u ranging from a velocity of the order of the ion thermal velocity $u_{Ti} = \sqrt{T_i/M}$ to the ion acoustic velocity $u_0 = \sqrt{T/M}$ (cf. [1]). In general, the feedback effect of the ion-acoustic waves leads to the formation of a plateau on the particle distribution function.^[3] In what follows we neglect the diffusion of the heavy ions under the effect of the waves and assume, for simplicity, that the ions are cold, $T_i = 0$ (so that $u_{Ti} = 0$). We assume that the plateau on the electron distribution function is formed so rapidly that it is in existence at any instant of time (cf. Appendix); our problem then is to analyze the time variation of the electron distribution function.

In the coordinate system in which the mean electron velocity $\bar{v}$ vanishes (Fig. 1) the ion distribution function and the associated plateau on the electron distribution function execute periodic motion with amplitude $V_0 = eE_0/m\omega$ (the field amplitude E_0 is limited by the condition $V_0 < v_{Ti}$; if this condition is not satisfied, the growth rate is comparable with the frequency and the quasi-linear theory no longer applies). Assume that at time t the ions as a whole have a velocity $V = V_0 \sin \omega t$

FIG. 1. Electronic (f) and ionic (solid vertical line) distribution functions. The plateaus of width u_0 execute periodic motion in the region $(-V_0, V_0)$. The case shown in the figure corresponds to plateau motion towards increasing v .



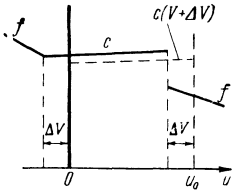


FIG. 2. Smoothing of the distribution function in the region $(0, u_0)$.

and that the height of the plateau on the electron distribution function $f(v) = c$ in the range $(V, V + u_0)$; in the ion coordinate system $(v = V + u)$ we have $f(u) = c$ in the range $0 < u < u_0$ (u is the statistical velocity of the electron in the ion coordinate system) (Fig. 2). In a time Δt the external electric field shifts the plateau by an amount $\Delta V = em^{-1}E_0 \cos \omega t \cdot \Delta t$ and the electron distribution function in the plateau region acquires the step shape shown in Fig. 2 by the solid line. The number of electrons in the plateau region is then reduced by $|\Delta V| [c - f(u_0)]$. On the other hand, this reduction is also equal to $u_0 [c(V) - c(V + \Delta V)]$, where $c(V + \Delta V)$ is the height of the plateau established after electron diffusion due to the ion-acoustic waves has occurred (the value $c(V + \Delta V)$ is shown in Fig. 2 by the dashed curve). Equating these expressions we find

$$\pm u_0 dc/dV = f(u_0 + V) - c(V), \quad (1)$$

where the plus sign is written when the plateau moves in the direction of increasing V (Figs. 1 and 2 illustrate this case) and the minus sign is written for motion in the opposite direction. It is evident from Fig. 2 that as the plateau moves there is a discontinuity at the fixed end and that beyond this region a continuous trail $f(V) = c$ remains.

We now consider the case of a strong electric field $u_0 \ll V_0$. If the width of the plateau u_0 is small it is evident from (1) that the distribution function $f(v)$ does not change appreciably in one period. Hence, from (1) we obtain an equation that describes the variation of $f(v)$ in a time much greater than the period of the external electric field $2\pi/\omega$.

We consider the time interval $\Delta \tau \sim 2\pi n/\omega$, $n \gg 1$, and find the quantity

$$\frac{\partial f}{\partial \tau} = \frac{\Delta f}{\Delta \tau} = \frac{(\Delta f)_n}{2\pi n/\omega},$$

where $(\Delta f)_n$ is the change in the distribution function in n periods as found from Eq. (1). The solution of Eq. (1) is

$$c = f \mp u_0 \partial f / \partial V + u_0^2 \partial^2 f / \partial V^2 \mp \dots = L_{\pm} f, \\ L_{\pm} = 1 \mp u_0 \frac{\partial}{\partial V} + \left(u_0 \frac{\partial}{\partial V} \right)^2 \mp \dots = \frac{1}{1 \pm u_0 \partial / \partial V}. \quad (2)$$

Thus, after a single traversal of the smoothing region the electron distribution function f is transformed into $L_{\pm} f$. In a full period the plateau moves in the forward and reverse directions so that $f \rightarrow L_{-}(L_{+} f) = (L_{-} L_{+}) f$ and in n periods $f \rightarrow (L_{+} L_{-})^n f$; the change $(\Delta f)_n$ of the distribution function f in n periods under the effect of the smoothing region of width u_0 is

$$(\Delta f)_n = (L_{+} L_{-})^n f - f = [(L_{+} L_{-})^n - 1] f.$$

Substituting this expression in the relation $\partial f / \partial \tau = (\Delta f)_n \omega / 2\pi n$, to accuracy of order $(u_0/V_0)^2$ we obtain the equation for the distribution function:

$$\partial f / \partial \tau = D \partial^2 f / \partial v^2, \quad D = u_0^2 \omega / 2\pi. \quad (3a)$$

Since the number of particles in the interval $(-V_0, V_0)$ is conserved the boundary condition for the diffusion equation (3a) is

$$(\partial f / \partial v)_{v=\pm V_0} = 0. \quad (3b)$$

Up to this point we have considered the diffusion of electrons due to waves whose phase velocities (in the ion coordinate system) lie in the range $(0, u_0)$. However, ion-acoustic waves are also excited in the range $(-u_0, 0)$. The plateaus on the electron distribution function f are different in these regions since the waves are damped on ions located at zero (we recall that the ions are cold $u_{Ti} = 0$; when $u_{Ti} \neq 0$ the plateau regions are separated by a space $|u| < u_{Ti}$ in which there are no waves because of strong damping on the ions). Hence, there are two independent smoothing regions, each of width u_0 , rather than one region of width $2u_0$; thus, the diffusion coefficient D is twice as large as that given in Eq. (3a):

$$D = u_0^2 \omega / \pi = \omega T / \pi M. \quad (3c)$$

The electron diffusion mechanism in the range $(-V_0, V_0)$ can be understood as follows. Let us consider an insulated rod of length $2V_0$ in which the thermal conductivity is infinitely large in some region u_0 ($u_0 < 2V_0$) and zero elsewhere. The temperature f is independent of distance v (measured from the center of the rod) in the region u_0 and is equal to c ; the temperature distribution f is maintained in time $f = f(v)$ in the remaining parts of the rod. If the region u_0 with infinite thermal conductivity is displaced by a distance ΔV (Fig. 2), the region beyond remains continuous $f(v) = c$ while the temperature c is changed in such a way that the amount of heat Q in the region u_0 is conserved ($Q = \int_0^{u_0} f dv = \text{const}$; the specific heat of the rod is taken as unity). Under these

conditions the temperature c in the region u_0 is described by Eq. (1).

If $u_0 \ll V_0$ the temperature f does not change appreciably in a single traversal of the region u_0 along the rod; in this case the slow equalization of the temperature f in the rod can be described by the "heat conductivity equation" (3a) and (3b) in which $\omega/2\pi = \varphi$ is the transit time of the rod through the region u_0 in the forward and reverse directions (the motion of the region in time can be arbitrary and can vary from one period φ to another; we consider the case $V(t) = V_0 \sin \omega t$). If the rod contains two regions of infinite thermal conductivity separated by some range in which the thermal conductivity is zero (Fig. 1) the "thermal-conductivity coefficient" V is given by Eq. (3c). Finally, the width of the smooth region u_0 can depend on a function $\{f\}$, for example

$$\{f\} = \int_{-V_0}^{V_0} \frac{mV^2}{2} f dV,$$

where m is a constant.

An exact solution of Eqs. (3a)–(3c) that takes account of the time variation of the temperature t and the diffusion coefficient D is given in the Appendix. We present the basic results (for the case in which the initial electron distribution function is Maxwellian).

In accordance with Eqs. (3a)–(3c) the distribution function $f(v)$ is equalized in the range $(-V_0, V_0)$ (Fig. 3) as $\tau \rightarrow \infty$, where

$$f_\infty = \frac{1}{2V_0} \int_{-V_0}^{V_0} f_0 dv. \quad (4a)$$

The characteristic equilibration time t_D is

$$t_D \sim V_0^2/D \sim V_0^2/\omega u_0^2. \quad (4b)$$

The electron temperature is increased by the following amount by virtue of the work done by the external electric field:

$$\Delta T = \int_{-V_0}^{V_0} \frac{mv^2}{2} [f_\infty - f_0] dv \quad (4c)$$

(the function f is normalized to unity).

The electron temperature increment ΔT grows as the amplitude of the electron oscillations V_0

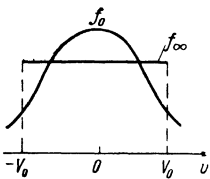


FIG. 3. Smoothing of distribution function in the interval $(-V_0, V_0)$ as $\tau \rightarrow \infty$.

increases and, for a fixed frequency ω , is proportional to the amplitude of the external field E_0 . If the amplitude of the electric field E_0 is such that V_0 is comparable with the electron thermal velocity $V_0 \sim V_T$ the change ΔT is comparable with T , that is to say, the electron temperature T increases by approximately a factor of two by virtue of the work done by the external electric field.

It is of interest to compare the increase in energy density ($n\Delta T$) with the Joule heat generated in the same time t_D . If the frequency of the external electric field ω is much larger than the collision frequency ν , the Joule heat generated per unit time is

$$Q \sim E_0^2 \omega \epsilon'' \sim E_0^2 \nu \omega_0^2 / \omega^2,$$

since $\epsilon'' = \text{Im } \epsilon = \text{Im} [1 - \omega_0^2 / \omega (\omega + i\nu)]$. When $V_0 \sim V_T$ the temperature change $\Delta T \sim T$ so that $n\Delta T / t_D Q \sim 1 / t_D \nu$.

These estimates show that the Joule heating can be much smaller than the collisionless heating if the equilibration time t_D is much smaller than the collision time $1/\nu$, that is to say, if the following inequality is satisfied:

$$1 \ll \frac{m}{M} \frac{\omega}{\omega_0} N_D \frac{1}{\lambda},$$

where N_D is the number of particles in a Debye sphere and λ is the Coulomb logarithm ($\lambda \sim 10$).

We have assumed up to this point that the electrons acquire a periodic velocity with respect to the ions in the external electric field. However, the motion of the electrons with respect to the ions can also result from electric fields in a wave of arbitrary amplitude which might be formed in the plasma. Such a wave must be rapidly damped because of the excitation of ion-acoustic waves and because of electron heating. We use energy conservation to estimate the damping time t_E for such a wave:

$$E_0^2(\tau)/8\pi + nT(\tau) + \mathcal{E}(\tau) = \text{const}. \quad (5)$$

Here, E_0 is the amplitude of the electric field of the wave in which the electrons acquire a velocity with respect to the ions, T is the electron temperature, $\mathcal{E}$ is the energy of the ion-acoustic waves. Neglecting the change in the latter (an appropriate estimate is given in the Appendix) we have from Eq. (5)

$$t_E \sim - \frac{E_0^2}{8\pi} \left/ \frac{d}{d\tau} \left(\frac{E_0^2}{8\pi} \right) \right. \sim \frac{E_0^2/8\pi}{nT} \frac{T}{dT/d\tau} \sim \frac{E_0^2/8\pi}{nT} t_D.$$

We have shown above that the amplitude E_0 is limited by the inequality $V_0 = eE_0/m\omega < V_T$; when $V_0 \sim V_T$ the damping time is given by

$$t_E \sim \frac{\omega^2}{\omega_0^2} t_D \sim \frac{1}{\omega} \left(\frac{\omega}{\omega_{0i}} \right)^2. \quad (6)$$

When $\omega \sim \omega_{0i}$ the wave is damped in a time of the order of one period $1/\omega$. This damping mechanism was invoked^[3] in the interpretation of experiments carried out by Zavoiskii et al^[4] (in which connection the present problem was studied). However, in these experiments the amplitude of the field E_0 was so large that the maximum electron velocity with respect to the ions V_0 was greater than the electron thermal velocity v_T . Under these conditions the waves excited in the plasma have growth rates comparable with the frequency, so that the damping time and increase in electron temperature given above can differ from the corresponding experimental quantities.

The author is indebted to A. A. Vedenov for suggesting this problem and for guidance and to M. A. Leontovich for valuable discussions.

APPENDIX

The derivation of Eq. (1) is based on the assumption that the plateau in the distribution function exists at any time in each diffusion region $(-u_0, 0)$ and $(0, u_0)$. This assumption is justified if there are always ion-acoustic waves with phase velocities u in these regions and if the wave energy is large enough so that the characteristic electron diffusion time $\langle t \rangle$ (for the establishment of the plateau) in velocity space is much smaller than the period of the external electric field.

We first investigate the way in which the energy of the ion acoustic-waves changes in motions of the plateau region. The equations of the quasi-linear theory, which describe the diffusion of electrons due to wave effects in the region $0 < u < u_0$, shown in Fig. 2, are of the form:^[3]

$$\begin{aligned} \partial \varepsilon / \partial t &= A \varepsilon \partial f / \partial u, & \varepsilon &= E^2 / 8\pi, \\ \partial f / \partial t &= \partial (\mathcal{D} \partial f / \partial u) / \partial u, & \mathcal{D} &= B \varepsilon. \end{aligned} \quad (A.1)$$

The functions A and B in (A.1) depend only on the velocity u . If the ion-acoustic waves are characterized by the dispersion relation

$$\omega^2 = \omega_{0i}^2 k^2 / (k^2 + D_e^{-2})$$

the functions A and B are given by

$$\begin{aligned} A &= \pi \omega_{0e} u |u| \left[1 - \frac{u^2}{u_0^2} \right]^{1/2}, & B &= \frac{4\pi (e/m)^2}{|u| (1 - u^2/u_0^2)}, \\ AB^{-1} &= \left(\frac{m}{2e} \right)^2 \omega_{0e} u^3 \left(1 - \frac{u^2}{u_0^2} \right)^{3/2}. \end{aligned} \quad (A.2)$$

Here, ω_{0i} and ω_{0e} are the ion and electron plasma frequencies, D_e is the Debye radius, $u_0 = \sqrt{T/M}$ is the ion-acoustic velocity.

The equations in (A.1) have the integral

$$\frac{\partial}{\partial u} A^{-1} B \frac{\partial \varepsilon}{\partial t} - \frac{\partial f}{\partial t} = 0. \quad (A.3)$$

We introduce below in place of the time t the variable $V = V_0 \sin \omega t$. Inasmuch as there is a plateau in the region $(0, u_0)$, $f(u, V) = c(V)$ (Fig. 2), after integration of (A.3) over velocity from 0 to u we have (taking account of the fact that $\varepsilon_{u=0} = 0$ because of ion damping)

$$A^{-1} B \partial \varepsilon / \partial V = u dc / dV. \quad (A.4)$$

Hence

$$\varepsilon - \varepsilon_0 = u A B^{-1} (c - c_0). \quad (A.5)$$

The relation in (A.4) can also be obtained by integrating (A.3) over velocity between the limits $(u, u_0 + 0)$. Since $\varepsilon_{u=u_0+0} = 0$, we have

$$A^{-1} B \frac{\partial \varepsilon}{\partial V} = - \int_u^{u_0+0} \frac{\partial f}{\partial V} du. \quad (A.6)$$

The function f has a discontinuity at the point u_0 :

$$f(u, V) = c + (f - c) \theta(u - u_0 + V(t) - v), \quad (A.7)$$

where $\varphi(x) = 1$ when $x > 0$, $\varphi(x) = 0$ when $x < 0$; v is the ion coordinate in the coordinate system in which the electrons are at rest. Taking account of (1), from (A.6) and (A.7) we have

$$\begin{aligned} A^{-1} B \frac{\partial \varepsilon}{\partial V} &= - \int_u^{u_0+0} \frac{\partial f}{\partial V} du - \int_{u_0-0}^{u_0+0} \frac{\partial f}{\partial V} du \\ &= -(u_0 - u) \frac{dc}{dV} - (f - c) = u \frac{dc}{dV}. \end{aligned}$$

For motion of the region $(0, u_0)$ in the direction of lower V in place of (A.5) we have

$$\varepsilon - \varepsilon_0 = (u - u_0) A B^{-1} (c - c_0). \quad (A.8)$$

We introduce the notation

$$y = u_0 |A B^{-1}|, \quad z = |u A B^{-1}|. \quad (A.9)$$

Then, (A.5) and (A.6) can be written in the form

$$\varepsilon - \varepsilon_0 = (c - c_0) \times \begin{cases} z \\ -(y - z) \end{cases}, \quad (A.10)$$

where the upper factor is written for motion of the region in the direction of increasing V while the lower is written for motion in the direction of decreasing V . For the region $(-u_0, 0)$ the factors in (A.10) must be taken locally. Now, using (A.10) we must find an equation describing the change in wave energy after long intervals of time in the same way as (3) is obtained from (1). For this purpose, using (A.10) we compute the change of energy $\Delta \varepsilon$ in a time $\Delta \tau$ which extends over many periods of the external electric field (Fig. 4).

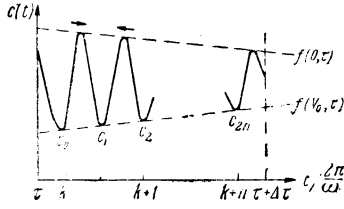


FIG. 4. Time dependence of the plateau level for the case of an even distribution function $f(v) = -f(v)$. The arrows indicate the direction of motion of the plateaus toward increasing v (the plateau level varies from c_0 to c_1) and in the opposite direction.

According to (A.10) and Fig. 4, in which we have shown the time dependence of the level of the plateau in the region $(0, u_0)$, the change of ϵ in one period is

$$(\Delta\epsilon)_1 = z(c_1 - c_0) - (y - z)(c_2 - c_1) = (z - y/2)(c_2 - c_0),$$

while the change in n periods is

$$(\Delta\epsilon)_n = (z - y/2)(c_{2n} - c_0).$$

Since the difference $\epsilon_{\sim}$ between the highest and lowest values of the energy of any wave is finite [from (A.10)] it is evident that $\epsilon_{\sim} = y[f(0, \tau) - f(V_0, \tau)]$, if n is high enough the changes of ϵ in the interval from τ to $2\pi k/\omega$ and from $2\pi(k+n)/\omega$ to $\tau + \Delta\tau$ can be neglected:

$$\frac{(\Delta\epsilon)_n}{2\pi n/\omega} \rightarrow \frac{\partial\epsilon}{\partial\tau}.$$

Substituting the expression for $(\Delta\epsilon)_n$ and taking account of the fact that

$$c_{2n} - c_0 = \Delta f(V_0, \tau) \left[1 + O\left(\frac{1}{n}\right) \right],$$

$$2\pi n/\omega = \Delta\tau \left[1 + O\left(\frac{1}{n}\right) \right],$$

we have

$$\frac{\partial\epsilon}{\partial\tau} = \left(z - \frac{y}{2} \right) \frac{\partial f(V_0, \tau)}{\partial\tau}. \quad (\text{A.11})$$

According to (A.2) and (A.9) the quantities y and z are weak functions of the time τ since $u_0 = u_0(\tau) = \sqrt{T(\tau)/M}$. In carrying out an estimate we neglect this dependence. From (A.11) we have

$$\epsilon(u, \tau) - \epsilon(u, 0) = (z - y/2) [f(V_0, \tau) - f(V_0, 0)]. \quad (\text{A.12})$$

Using the last relation we can determine when the wave existence criterion $\epsilon(u, \tau) > 0$ is satisfied in the regions $0 < |u| < u_0$. Since the quan-

tity $f(V_0, \tau) - f(V_0, 0)$ increases monotonically during the electron diffusion time, it follows from (A.9) and (A.12) that waves with phase velocity u in the range $0 < |u| < u_0/2$ will be damped. This condition imposes a limitation on the initial spectrum of excited waves $\epsilon(u, 0)$:

$$\epsilon(u, \infty) = \epsilon(u, 0) + (z - y/2) [f_{\infty} - f(V_0, 0)] > 0, \quad 0 < |u| < u_0/2. \quad (\text{A.13})$$

The last condition on $\epsilon(u, 0)$ is satisfied, for example, when the external electric field grows to an amplitude E_0 in several periods as shown in Fig. 5 (the electron diffusion during the growth time is neglected).

Since this process is analogous to that shown in Fig. 4, we can apply (A.12) taking account of the notation used in Fig. 5 [at the beginning of the process, $f = f(0, 0)$, $\epsilon = 0$ and at the end $f = f(V_0, 0)$, $\epsilon = \epsilon(u, 0)$]. As a result we have

$$\epsilon(u, 0) = \left(z - \frac{y}{2} \right) \times \begin{cases} f(V_0, 0) - f(0, 0), & 0 < |u| < u_0/2 \\ 0, & u_0/2 < |u| < u_0 \end{cases} \quad (\text{A.14})$$

so that in the electron diffusion process, in accordance with (A.12),

$$\epsilon(u, \tau) = \left(z - \frac{y}{2} \right) \times \begin{cases} f(V_0, \tau) - f(0, 0), & 0 < |u| < u_0/2 \\ f(V_0, \tau) - f(V_0, 0), & u_0/2 < |u| < u_0 \end{cases} \quad (\text{A.15})$$

Using (A.1) and (A.2) we can write the second criterion for the existence of a plateau in the form

$$1/\omega \gg \langle t \rangle \sim u_0^2 / \langle B \rangle \langle \epsilon \rangle. \quad (\text{A.16})$$

For the initial distribution $\epsilon(u, 0)$ given in (A.14) the characteristic energy of the ion-acoustic waves $\langle \epsilon \rangle$ is taken to be

$$\langle \Delta\epsilon \rangle \sim \langle y \rangle [f(0, 0) - f(V_0, 0)] \sim \langle y \rangle / v_{Te}, \quad (\text{A.17})$$

which, taking account of (A.9) yields

$$1/\omega \gg v_{Te} / \omega_{0e} u_0 \sim 1/\omega_{0i}.$$

If, however, $\langle \epsilon(u, 0) \rangle \gg \langle \Delta\epsilon \rangle$, in the estimate in (A.16) we must take $\langle \epsilon \rangle = \langle \epsilon(u, 0) \rangle$.

We now estimate the total change of wave energy. This is obtained by integrating the right side of (A.12) over k between the limits $(-\infty, \infty)$. Since y and z are functions of the phase velocity u it is convenient to convert to the integration variable u , which is expressed in terms of k by the dispersion relation $u^2 = \omega_{0i}^2 / (k^2 + D_e^2)$. We have

$$\begin{aligned} \Delta\mathcal{E}(\tau) &= -2 \int_0^{u_0} \frac{\omega_{0i} du}{u^2 \sqrt{1 - u^2/u_0^2}} \left(z - \frac{y}{2} \right) [f(V_0, \tau) - f(V_0, 0)] \\ &\sim \frac{\omega_{0i}}{u_0} \langle \Delta\epsilon \rangle \sim nT \left(\frac{m}{M} \right)^2. \end{aligned}$$

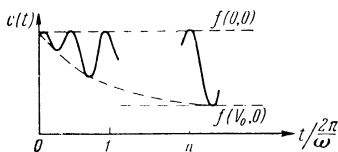


FIG. 5. Variation of the plateau level in the case of a gradual increase in the field amplitude to the stationary value.

We can now compare the energy changes in (5):

$$\Delta\left(\frac{E_0^2}{8\pi}\right): (n\Delta T): \Delta\mathcal{E} \sim \frac{E_0^2}{8\pi}: nT \frac{t_E}{t_D}: \Delta\mathcal{E} \sim 1:1: \left(\frac{m}{M} \frac{\omega_{0e}}{\omega}\right)^2. \quad (\text{A.18})$$

In conclusion, we write the solution of (3) taking account of the time variation of the diffusion coefficient. Let us start with the equation

$$\partial f / \partial \tau = D \partial^2 f / \partial v^2 \quad (\text{A.19})$$

with initial and boundary conditions in which D depends on the functional $\{f\}$ of a function to be determined so that it is an unknown function of time. The form of the functional $\{f\}$ and the function $D = D(\{f\})$ are known.

Replacing the time τ by the variable x

$$dx = D d\tau, \quad x_{\tau=0} = 0, \quad (\text{A.20})$$

we obtain from (A.19) an equation with exactly the same conditions, but for which $D = 1$. If it is possible to write the solution of the equation $f(x, v)$ in explicit form then $D(x) = D(\{f(x, v)\})$ is also known, and after substitution in (A.2) we have

$$\tau = \int_0^x \frac{dx'}{D(\{f(x, v)\})}. \quad (\text{A.21})$$

The function to be determined $f(\tau, v)$ is found by eliminating the parameter x from (A.21) and the relation $f = f(x, v)$.

For the equations (3a)–(3c) we have

$$\begin{aligned} D &= \frac{\omega}{\pi} \frac{T(\tau)}{M} = \frac{\omega}{\pi M} (T_0 + \{f\} - \{f_0\}), \\ \{f\} &= \int_{-V_0}^{V_0} \frac{mv^2}{2} f dv, \quad T_0 = T(0), \quad f_0 = f(0, v), \\ f(x, V) &= \sum_{n=0}^{\infty} f^{(n)} \exp\left\{-\frac{\pi^2}{V_0^2} n^2 x\right\} \cos n\pi \frac{v}{V_0}, \\ f_0(v) &= \sum_{n=0}^{\infty} f^{(n)} \cos n\pi \frac{v}{V_0}, \\ \tau &= \frac{\pi M}{\omega} \int_0^x \left[T_0 + \sum_{n=1}^{\infty} f^{(n)} \left(\exp\left\{-\left(\frac{\pi n}{V_0}\right)^2 x'\right\} - 1 \right) m V_0^3 \frac{(-1)^n}{(n\pi)^2} \right]^{-1} dx'. \end{aligned}$$

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CONCENTRATION OF A LIGHT WAVE IN A WEAKLY INHOMOGENEOUS DIELECTRIC

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We consider the transmission of a plane wave through a smoothly stratified inhomogeneity. At angles of incidence close to that corresponding to total reflection from the layer the energy density inside the inhomogeneity can become many times greater than the mean energy density in the wave. The possible connection between this effect and the granulated structure of radiation from laser crystals is discussed.

THE propagation of waves in inhomogeneous media has been treated by a number of workers.^[1-3] It is the purpose of the present work to call attention to the fact that an appreciable field concentration is possible even when the inhomogeneity is weak ($\Delta n/n \ll 1$). The field concentration effect is usually accompanied by the generation of heat which, in turn, causes a change in the optical inhomogeneity. This effect may be responsible for the field formation in crystals used in lasers. We shall treat the problem of field concentration in a wave transmitted through a weak optical inhomogeneity in the simplest possible terms.

Assume that in an infinite nonabsorbing non-magnetized isotropic medium there is a smooth plane stratified inhomogeneity on which, at an angle θ , is incident a plane polarized monochromatic wave. We compute the ratio of the energy density inside the stratification to the energy far from the stratification as a function of θ . The electric field E_θ is assumed to be perpendicular to the plane of incidence. Taking the y axis parallel to E_θ and the z axis perpendicular to the stratification, we write the following equations for the complex field amplitudes:

$$\partial^2 E_\theta / \partial x^2 + \partial^2 E_\theta / \partial z^2 + k^2(z) E_\theta = 0, \quad k(z) = \omega c^{-1} \sqrt{\epsilon(z)}.$$

The quantity $E_\theta(x, z)$ is then given by

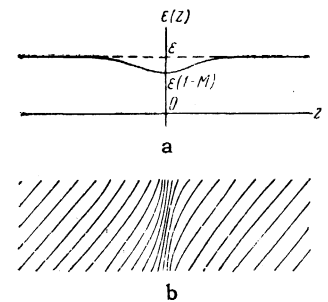
$$E_\theta(x, z) = X_\theta(x) Z_\theta(z), \quad X_\theta(x) = X \exp(ikx \sin \theta), \\ d^2 Z_\theta / dz^2 + [k^2(z) - k^2 \sin^2 \theta] Z_\theta = 0, \quad k = \omega c^{-1} \sqrt{\epsilon},$$

where ϵ and θ are the values of the dielectric constant of the medium and the angle of incidence at far distances from the stratification.

We shall assume that $\epsilon(z)$ is given by the Epstein function for a symmetric stratification^[1,2] (Fig. 1a):

$$\epsilon(z) = \epsilon [1 - 4Me^{mz} / (1 + e^{mz})^2].$$

FIG. 1. The variation of $\epsilon(z)$ in a symmetric Epstein layer (a) and refraction in an Epstein layer (b).



We now compute the quantity

$$Q(\theta, z) = \epsilon(z) |E_\theta(z)|^2 / \epsilon |E_\theta(-\infty)|^2$$

which is the ratio of energy density inside the stratification to that in the incident wave far from the stratification.¹⁾ When

$$Z_\theta = F_\theta(\xi) \xi^{1/2} e^{iS \cos \theta} (1 - \xi)^\alpha, \quad \xi = -e^{mz}$$

the function $F_\theta(\xi)$ satisfies the hypergeometric equation

$$\frac{d^2 F_\theta}{d\xi^2} - \frac{(2\alpha + iS \cos \theta + 1)\xi - (1 + iS \cos \theta)}{\xi(1 - \xi)} \frac{dF_\theta}{d\xi} - \frac{\alpha(\alpha + iS \cos \theta)}{\xi(1 - \xi)} F_\theta = 0,$$

$$\alpha = 1/2 (1 + \sqrt{1 - 4S^2 M}), \quad S = 2k/m.$$

The parameter S and the stratification thickness are related by the expression $S \cong 4l/\lambda$ where λ is the wavelength far from the inhomogeneity.

¹⁾The ratio of the energy density inside the inhomogeneity to the mean value far from the inhomogeneity is affected by the reflected wave as well as the incident wave. It differs from Q by the factor

$$R(\theta) = \{1 + \exp[2\pi S(\cos \theta - \sqrt{M})]\} / \{2 + \exp[2\pi S(\cos \theta - \sqrt{M})]\},$$

which ranges between $1/2$ and 1 .

The value of $Q(\theta, z)$ at the center of the stratification is

$$Q(\theta, 0) = \frac{1-M}{4\pi} \left| \frac{\Gamma(1/4 + 1/4 \sqrt{1-4A^2} - 1/2 iA\tau) \Gamma(1/4 - 1/4 \sqrt{1-4A^2} - 1/2 iA\tau)}{\Gamma(iA\tau)} \right|^2,$$

where

$$A = S\sqrt{M}, \tau = M^{-1/2} \cos \theta.$$

We consider briefly the case $M > 0$ for which the optical density in the inhomogeneous stratification is smaller than that of the uniform medium, assuming that the density in the stratification does not differ greatly from that of the medium but that the relative thickness of the stratification is large: $M \ll 1$, $A \gg 1$. In this case, for angles that are not too close to the angle corresponding to total reflection $\theta_0 = \cos^{-1} \sqrt{M}$ we have $Q(\theta, 0) \cong \cos \theta / \sqrt{|\cos^2 \theta - M|}$. This formula can be easily obtained from geometric optics: it describes the concentration of energy at the center of the inhomogeneity due to the refraction and concentration of rays in the medium of lower density (Fig. 1b).

At angles close to θ_0 , we find $Q(\theta, 0) \cong \sqrt{A} \times \exp[-2S^2(\theta - \theta_0)^2]$, that is to say, when $|\theta - \theta_0| < 1/2S$ the energy density at the center of the inhomogeneity is $\sqrt{A}$ times greater than the mean value (Fig. 2).

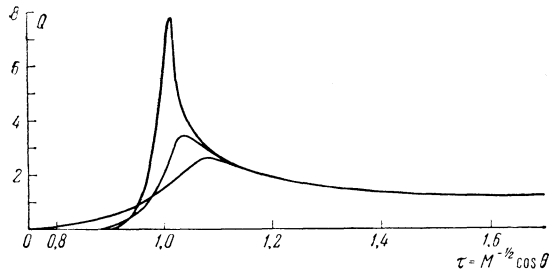


FIG. 2. Field concentration as a function of angle of incidence at the stratification; $A = 100; 10; 5$ (in order of decreasing maxima).

If $M < 0$ a region of guided propagation is produced in the dielectric. We shall not consider this case here. It is our opinion that as far as lasers are concerned the region of guided propagation is not of great interest since it is isolated from the remaining medium and is thus not convenient for generation.

The estimate $Q(\theta_0, 0) \cong \sqrt{A}$ is obtained under several simplifying assumptions, one of which is that $\epsilon(z)$ is real at the frequency being considered. Absorption in a solid dielectric means, first of all, that there will be enhanced heat generation at points of local field concentration; the local heating can have a marked effect on the inhomogeneity.

For example, if field concentration in a dielectric that is initially uniform causes a heating of several degrees, an inhomogeneity will be formed (in the heated channel) for which the value $|M| \cong \epsilon^{-1} \Delta T |d\epsilon'/dT|$ is $10^{-3}-10^{-4}$; if $d\epsilon'/dT < 0$, $l \cong 5 \times 10^{-2}$ cm, and $\lambda = 8 \times 10^{-5}$ cm we have $Q(\theta_0, 0) \cong 5-10$. Depending on the sign of the thermal effect the field concentration of the light wave can be strengthened or weakened.

These considerations also apply to a laser crystal. In order to obtain a negative balance between absorption and amplification one usually chooses a material in which the absorption is a minimum; at the generation frequency; the primary source of heating here is the loss due to the transformation of the acquired energy in the generated radiation. For example, to obtain photons at the generation frequency one must use radiationless transitions in a ruby laser. A simple calculation shows that an increase in the intensity of induced transitions means that there is an increase in the probability for spontaneous radiationless transitions; hence, field concentration of the generated radiation leads to local heating. The sign of $d\epsilon'/dT$ determines the thermal stability of the uniformity of the field associated with the light wave; since generation occurs in a region of anomalous dispersion the sign of $d\epsilon'/dT$ can depend on frequency.

We consider briefly the field distribution in a laser crystal. Usually the mirrors at the ends are perpendicular to Ox —the direction of the optic axis. One or more discrete modes, characterized by the wave vectors $\mathbf{k}$ participate in the oscillation. As a consequence of fluctuations the wave vectors for each mode are smeared to some extent both in magnitude and direction. In a cylindrical crystal the highest Q will be that of the axial modes whose wave vectors make small angles with the Ox axis. The inhomogeneity of the field in the axial modes can only have a stratified structure since any inhomogeneity in the Ox direction is rapidly equalized as a result of induced radiation effects. We use the notation $(y_1, z_1), (y_2, z_2), \dots$ to denote segments parallel to Ox along which the optical path between mirrors is a minimum. These segments surround regions of lower optical density in which the field is concentrated. Thus, at small intensities individual saturation “wavelets” $(y_1, z_1), (y_2, z_2), \dots$ are generated; the greater bulk of the crystal remains essentially in the linear amplification mode. Each mode corresponds to an

interference pattern determined by the geometry of the crystal and the distribution of $\epsilon(y, z)$. The radiation of all the small portions of one mode appearing as maxima in this pattern is coherent.

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SMEARING OUT OF INHOMOGENEITIES IN A WEAKLY IONIZED PLASMA IN A MAGNETIC FIELD (AMBIPOLAR DIFFUSION)

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It is shown that the process of smearing out of an inhomogeneity in a plasma, called ambipolar diffusion, is of a complex nature in a magnetic field and cannot be described by the normal diffusion equation. The velocity of the inhomogeneity across the magnetic field depends significantly on the initial size of the inhomogeneity and may be many times larger than the transverse electron diffusion velocity.

DURING the smearing out of inhomogeneities in a plasma, a difference in electron and ion concentration arises as a result of the different velocities of these particles. The changes in concentration, however, produce an electric field which inhibits further separation of the electrons and ions. As a result, the inhomogeneity is "smoothed out" in such a manner that the electron and ion concentrations are always approximately equal to each other. This process is also called ambipolar diffusion. Schottky^[1] showed that in the absence of a magnetic field ambipolar diffusion resembles the normal diffusion process and proceeds at the rate of the normal ion diffusion velocity multiplied by a factor $1 + T_e/T_i$, owing to the effect of the electrons.

In the presence of a magnetic field the diffusion becomes anisotropic. The electrons, as before, diffuse more rapidly along the magnetic field lines, so that ambipolar diffusion in this direction has the velocity of the slower particles, the ions. Across the magnetic field, on the other hand, the ions diffuse more rapidly; ambipolar diffusion in a strictly transverse direction also takes place at the speed of the slowest particles, the electrons¹⁾ multiplied by $1 + T_i/T_e$ on account of the effects of the ions^[2]

How then is an inhomogeneity of arbitrary form smeared out? It would seem that knowing the diffusion velocity in the transverse and longitudinal directions, one could readily derive the rules governing the smearing out of any inhomogeneity. However this kind of naive discussion, presented

in some works (see^[3]), is shown below to be invalid and leads to serious misrepresentations of the nature of the effect. An analysis of this very question is discussed in the present work.

We consider an inhomogeneity in a weakly ionized plasma (i.e., when an important role is played by collisions of electrons and ions with neutral molecules) situated in a uniform magnetic field. We assume that the inhomogeneity does not vary appreciably over the length of a mean free path and does not change significantly during the mean free time of the charged particles, so that it is possible to use the macroscopic equations for describing the particle motion in the plasma²⁾. These equations, after linearization, take the following form:

$$\partial \delta n_e / \partial t + n_0 \nabla \mathbf{v}_e = 0, \quad (1a)^*$$

$$\partial \delta n_i / \partial t + n_0 \nabla \mathbf{v}_i = 0, \quad (1b)$$

$$m \nu_{em} n_0 \mathbf{v}_e = -e n_0 \mathbf{E} - (e n_0 / c) [\mathbf{v}_e \mathbf{H}_0] - \kappa T_e \nabla \delta n_e, \quad (1c)$$

$$M \nu_{im} n_0 \mathbf{v}_i = e n_0 \mathbf{E} + (e n_0 / c) [\mathbf{v}_i \mathbf{H}_0] - \kappa T_i \nabla \delta n_i. \quad (1d)$$

Here δn_e (or δn_i) is the excess concentration of electrons (or ions) in the inhomogeneities, $n_0 = n_{e0} = n_{i0}$ is the mean electron and ion concentration ($\delta n \ll n_0$), e is the charge on the electron, m and M are the electron and ion masses respectively, κ is Boltzmann's constant, T_e and T_i are the electron and ion temperatures respectively, $\mathbf{E}$ is the electric field strength, and ν_{em} and ν_{im} are

²⁾On the basis of this condition inertial ($\partial \mathbf{v} / \partial t$) and viscous ($\eta \nabla^2 \mathbf{v}$) terms can be omitted from the macroscopic equation of motion for the electrons and ions, since they are small in comparison to the term representing direct retardation produced by molecules.

* $[\mathbf{v}_e \mathbf{H}_0] = \mathbf{v}_e \times \mathbf{H}_0$.

¹⁾Simon^[2] has shown that in the presence of metallic electrodes, increased diffusion of a plasma in the transverse direction is possible. In the present work we consider only an unbounded plasma.

the collision frequencies of the electrons and ions with neutral molecules (see, for example, [4]). Equations (1c) and (1d) apply to a system of coordinates in which the magnetic field H_0 is constant; the gas of neutral molecules is assumed to be at rest relative to the magnetic field³⁾.

The system (1) has to be completed by including the field equations. If the following condition is also satisfied,

$$(\nu\omega_0/c\nu)^2 \ll 1, \quad (2)$$

where ν is the thermal velocity, ν the collision frequency and ω_0 the Langmuir (plasma) frequency, then the effect of the solenoidal electric field on the motion of the charged particles can be neglected. In this case only the longitudinal electric field is of importance, and,

$$\Delta\varphi = 4\pi e (\delta n_i - \delta n_e). \quad (3)$$

Condition (2) is generally valid in a weakly ionized plasma; this is the case which will be considered below⁴⁾.

It is natural to seek the solution of the system of linear equations (1) and (3) by expanding the unknown functions in Fourier integrals with respect to the co-ordinates,

$$\delta n_{\mathbf{k}} = \int \delta n(\mathbf{r}) e^{-i\mathbf{k}\mathbf{r}} d^3r$$

etc. The equations for the Fourier components of the potential $\varphi_{\mathbf{k}}$ and the velocities $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ are algebraic. Therefore the functions $\varphi_{\mathbf{k}}$, $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ are expressible simply in terms of $\delta n_{\mathbf{k}e}$ and $\delta n_{\mathbf{k}i}$. Substituting the expressions for $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ into the equations for $\delta n_{\mathbf{k}e}$ and $\delta n_{\mathbf{k}i}$, we obtain

$$\begin{aligned} \partial \delta n_{\mathbf{k}e} / \partial t + \alpha_{He} [D_e k^2 \delta n_{\mathbf{k}e} + 4\pi\sigma_e (\delta n_{\mathbf{k}e} - \delta n_{\mathbf{k}i})] &= 0, \\ \partial \delta n_{\mathbf{k}i} / \partial t + \alpha_{Hi} [D_i k^2 \delta n_{\mathbf{k}i} + 4\pi\sigma_i (\delta n_{\mathbf{k}i} - \delta n_{\mathbf{k}e})] &= 0. \end{aligned} \quad (4)$$

In order to simplify the form of the equations, the following symbols have been introduced

$$\alpha_{He} = \frac{1 + (\omega_H/\nu_{em})^2 \cos^2 \beta}{1 + (\omega_H/\nu_{em})^2}, \quad \alpha_{Hi} = \frac{1 + (\Omega_H/\nu_{im})^2 \cos^2 \beta}{1 + (\Omega_H/\nu_{im})^2}.$$

³⁾We consider here isothermal diffusion. It is therefore permissible to ignore the motion of the neutral molecules in this case only when the inhomogeneities are not too large: i.e., $L^2 \ll l^2 (n_m/n_0)$, where L is the dimension of the inhomogeneity, l is the mean free path of the particles and n_m is the number density of the molecules.

⁴⁾We note that in a strongly ionized plasma, where the major role is played by collisions between the charged particles themselves, condition (2) is generally not satisfied and in this case one cannot neglect the influence of the solenoidal electric fields.

Furthermore $D_e = \kappa T/m\nu_{em}$, $\sigma_e = e^2 n_0/m\nu_{em}$ and $\omega_H = eH_0/mc$ are respectively the coefficients of longitudinal diffusion, the conductivity, and Larmor frequency for electrons, while D_i , σ_i and Ω_H are the corresponding quantities for the ions; β is the angle between $\mathbf{k}$ and $\mathbf{H}_0$.

The solution of the linear equations (4) can clearly be written in the form,

$$\begin{aligned} \delta n_{\mathbf{k}e}(t) &= \delta n_{\mathbf{k}e}^{(1)} e^{-q_1 t} + \delta n_{\mathbf{k}e}^{(2)} e^{-q_2 t}, \\ \delta n_{\mathbf{k}i}(t) &= \delta n_{\mathbf{k}i}^{(1)} e^{-q_1 t} + \delta n_{\mathbf{k}i}^{(2)} e^{-q_2 t}, \end{aligned} \quad (5)$$

where q_1 and q_2 are the roots of the characteristic equation. If the typical dimensions of the perturbed region are large compared with the Debye radius, R_D ,

$$(kR_D)^2 = k^2 \kappa T_e T_i / 4\pi e^2 n_0 (T_e + T_i) \ll 1, \quad (6)$$

then the expressions for q_1 and q_2 have the simple form:

$$\begin{aligned} q_1 &= 4\pi\sigma_e \alpha_{He} + 4\pi\sigma_i \alpha_{Hi}, \\ q_2 &= k^2 \frac{(\sigma_e D_i + \sigma_i D_e) \alpha_{He} \alpha_{Hi}}{\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}} = k^2 \frac{\kappa (T_e + T_i)}{M\nu_{im}/\alpha_{Hi} + m\nu_{em}/\alpha_{He}}. \end{aligned} \quad (7)$$

For $\delta n_{\mathbf{k}e}^{(1)}$, $\delta n_{\mathbf{k}e}^{(2)}$, $\delta n_{\mathbf{k}i}^{(1)}$, and $\delta n_{\mathbf{k}i}^{(2)}$, we obtain in this case

$$\begin{aligned} \delta n_{\mathbf{k}e}^{(1)} &= \frac{\sigma_e \alpha_{He}}{\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}} [\delta n_{\mathbf{k}e}(0) - \delta n_{\mathbf{k}i}(0)], \\ \delta n_{\mathbf{k}i}^{(1)} &= \frac{\sigma_i \alpha_{Hi}}{\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}} [\delta n_{\mathbf{k}i}(0) - \delta n_{\mathbf{k}e}(0)], \end{aligned} \quad (8a)$$

$$\delta n_{\mathbf{k}e}^{(2)} = \delta n_{\mathbf{k}i}^{(2)} = [\sigma_e \alpha_{He} \delta n_{\mathbf{k}i}(0) + \sigma_i \alpha_{Hi} \delta n_{\mathbf{k}e}(0)] / (\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}). \quad (8b)$$

Here $\delta n_{\mathbf{k}e}(0)$ and $\delta n_{\mathbf{k}i}(0)$ are the Fourier components of the initial perturbed density of the electrons and ions.

From (5), and (8) it is evident that the root q_1 corresponds to the dispersal of the initial charge in the plasma. The root q_2 corresponds to the smearing out of the inhomogeneity in such a manner that the concentrations of ions and electrons remain identical; this process can be called "ambipolar diffusion." As is known, the dispersal of excess charges proceeds at a much higher rate than diffusion: $q_1/q_2 \sim 1/(kR_D)^2$. Therefore for arbitrary initial electron and ion concentrations, there is established after a time $\tau \sim 1/q_1$ an inhomogeneity in which the concentrations δn_e and δn_i are identical (8b), following which the ambipolar diffusion commences. Here

$$\begin{aligned} \delta n(\mathbf{r}, t) &= (2\pi)^{-3} \int \delta n_{\mathbf{k}}(0) \exp \{i\mathbf{k}\mathbf{r} - D(\beta) k^2 t\} d^3k, \\ D(\beta) &= \kappa (T_e + T_i) \{M\nu_{im} [1 + (\Omega_H/\nu_{im})^2] / [1 + (\Omega_H/\nu_{im})^2 \cos^2 \beta] \\ &\quad + m\nu_{em} [1 + (\omega_H/\nu_{em})^2] / [1 + (\omega_H/\nu_{em})^2 \cos^2 \beta]\}^{-1}, \end{aligned} \quad (9)$$

where β is the angle between the wave vector $\mathbf{k}$ and the magnetic field, and $\delta n_{\mathbf{k}}(0)$ is the Fourier component of the inhomogeneity in the ion or electron density after the conditions described by (8b) have been established.

We note that if the smearing out of the inhomogeneity could be described by the usual diffusion equation, then the coefficient $D(\beta)$ would be represented by

$$D(\beta) = D_{\parallel} \cos^2 \beta + D_{\perp} \sin^2 \beta, \quad (10)$$

where $D_{\parallel}$ and $D_{\perp}$ are the coefficients of longitudinal and transverse diffusion. In our case the coefficient $D(\beta)$ cannot as a rule, be represented by such an expression, so that the smearing out process in a plasma, i.e., ambipolar diffusion, does not have the same characteristics as ordinary diffusion and is of a more complex nature; it also depends appreciably on the initial structure of the inhomogeneity.

In particular, if the inhomogeneity does not extend for any great length along the magnetic field, or more accurately, if

$$R_{\parallel}^2 \ll R_{\perp}^2 (Mv_{im}/mv_{em}) [1 + (\Omega_H/v_{im})^2], \quad (11)$$

where $R_{\parallel}$ and $R_{\perp}$ are the typical dimensions of the inhomogeneity in directions parallel and perpendicular to the magnetic field, then the second term in the denominator in the expression for $D(\beta)$ is small. In this case normal diffusion takes place [Eq. (10)] with,

$$D_{\parallel} = \kappa (T_i + T_e) / Mv_{im},$$

$$D_{\perp} = \kappa (T_i + T_e) / Mv_{im} [1 + (\Omega_H/v_{im})^2]. \quad (12)$$

It is readily seen that condition (11) holds during the whole development process of the inhomogeneity. Therefore if the initial inhomogeneity is such that the condition (11) is fulfilled, then ambipolar diffusion in both the longitudinal and transverse directions proceeds at the diffusion rate of the ions. The effect of the electrons is only to introduce an additional factor of $(1 + T_e/T_i)$.

If on the other hand the initial inhomogeneity extends over appreciable distances along the magnetic field, so that the reverse of condition (11) is satisfied, then diffusion proceeds in directions perpendicular to a strong magnetic field at a considerably reduced rate, that of the diffusion of the electrons:

$$D_{\perp} = \kappa (T_i + T_e) / (Mv_{im} + m\omega_H^2/v_{em}).$$

The diffusion proceeds along the magnetic field, as before, at the diffusion rate of the ions. We note

also that anisotropy in diffusion for very extensive inhomogeneities begins at weaker magnetic fields [when $H_0 > (Mm\nu_{im}\nu_{em}c^2)^{1/2}/e$], than for the reverse case (11) (when $H_0 > M\nu_{im}c/e$).

Thus, the smearing out of inhomogeneities in directions at right angles to the magnetic field in a plasma depends greatly on the form of the inhomogeneity and generally speaking, proceeds much faster than the transverse diffusion velocity of electrons. The balancing of the electron and ion concentrations in the region of the inhomogeneity in this case is achieved by the formation of current loops which are produced as a result of the convective motion of electrons along the lines of force of the magnetic field. If the inhomogeneity extends over large distances along the field, then the balancing process is complicated, and the diffusion of electrons transverse to the magnetic field begins to play a dominant role.

Since the smearing out process in a magnetic field is accompanied by the production of electric currents, magnetic perturbations are in turn produced by such currents. The Fourier components for the perturbed magnetic field have the form

$$\delta \mathbf{H}_{\mathbf{k}} = - \frac{4\pi e \delta n_{\mathbf{k}} (\sigma_e D_i + \sigma_i D_e)}{c (\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}) [1 + (\omega_H/v_{em})^2] [1 + (\Omega_H/v_{im})^2]} \\ \times \left\{ \left(\frac{\omega_H}{v_{em}} \right)^2 \cos \beta \left[\frac{\mathbf{k}}{k} \frac{\mathbf{H}_0}{H_0} \right] + \frac{\omega_H}{v_{em}} \left(1 + \frac{\omega_H \Omega_H}{v_{em} v_{im}} \cos^2 \beta \right) \left(\frac{\mathbf{H}_0}{H_0} - \frac{\mathbf{k}}{k} \cos \beta \right) \right\}.$$

In the case of longitudinal diffusion ($\mathbf{k} \parallel \mathbf{H}_0$) there are no perturbations in the magnetic field, as is to be expected.

The solution for ambipolar diffusion given above is only valid in a first approximation up to $(kR_D)^2$. In the next approximation the electron and ion concentrations no longer remain equal and the inhomogeneity is polarized to an extent represented by,

$$\Phi_{\mathbf{k}} = \frac{\kappa \delta n_{\mathbf{k}} T_e M v_{im} \alpha_{He} - T_i m v_{em} \alpha_{Hi}}{en_0 M v_{im} \alpha_{He} + m v_{em} \alpha_{Hi}}.$$

From this result it is evident that the potential of the electric field in the inhomogeneity reverses sign, depending on the shape of the inhomogeneity.

We also note that the system (1) and (3) can be reduced (if we restrict ourselves to inhomogeneities for which $\delta n_e \approx \delta n_i = \delta n$) to a single equation, provided condition (6) is valid:

$$(\sigma_e \Delta_{He} + \sigma_i \Delta_{Hi}) \frac{\partial \delta n}{\partial t} = (\sigma_e D_i + \sigma_i D_e) \Delta_{He} \Delta_{Hi} \delta n. \quad (13)$$

Here

$$\Delta_{He} = \partial^2 / \partial z^2 + [1 + (\omega_H/v_{em})^2]^{-1} (\partial^2 / \partial x^2 + \partial^2 / \partial y^2),$$

and Δ_{Hi} is an analogous operator except that ω_H/ν_{em} in it is replaced by Ω_H/ν_{im} . Equation (13) is also the equation of ambipolar diffusion in a weakly ionized plasma in a magnetic field. It goes without saying that the solution of (13) is identical to (9). In the absence of a magnetic field, Eq. (13) leads to the normal diffusion equation with a diffusion coefficient $D = (\sigma_e D_i + \sigma_i D_e)/(\sigma_e + \sigma_i)$, which agrees with Shottky's results^[1].

In conclusion the author expresses his thanks to V. L. Ginzburg and V. P. Silin for valuable discussions.

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NONLINEAR EFFECTS IN THE ELECTRODYNAMICS OF A TRANSPARENT MEDIUM

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Equations are deduced which describe the nonlinear interactions of electromagnetic waves in a transparent medium. The results are applied to the study of second harmonics arising in the propagation of the wave in the medium. The intensities and polarizations of the harmonics in quartz are determined.

1. Dynamic and kinetic equations describing nonlinear interactions between waves in a plasma, located in a strong magnetic field were obtained in [1]. In the present work, similar equations are obtained for an arbitrary medium.

Recently, the problems of the nonlinear electrodynamics of a medium have attracted interest in connection with the appearance of experimental possibilities for the study of nonlinear effects. For example, Franken et al [2] have reported that the appearance of a second harmonic was observed in the transmission of an intense monochromatic light beam in the optical frequency range through crystalline quartz. This phenomenon is very simply interpreted by means of the equations for wave interactions obtained in the present work.

2. For simplicity, we restrict ourselves to nonmagnetic media ($\mu \approx 1$) and write down Maxwell's equations in the form

$$\text{rot } \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}, \quad \frac{\partial \mathbf{H}}{\partial t} = -c \text{rot } \mathbf{E}. \quad (1)^*$$

The nonlinear effects are described by terms of second and higher orders in $\mathbf{E}$, depending on the polarization vector $\mathbf{P}$ and $\mathbf{E}$. By restricting ourselves to terms of second order, we can write (see [3])

$$\mathbf{P} = \mathbf{P}^{(1)} + \mathbf{P}^{(2)}; \quad (2)$$

$$P_{\alpha}^{(1)} = (4\pi)^{-1} \int_{-\infty}^t dt_1 \varphi_{\alpha\beta}^{(1)}(t - t_1) E_{\beta}(t_1), \quad (3)$$

$$P_{\alpha}^{(2)} = (4\pi)^{-1} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \varphi_{\alpha\beta\gamma}^{(2)}(t - t_1, t_1 - t_2) E_{\beta}(t_1) E_{\gamma}(t_2), \quad (4)$$

$\alpha, \beta, \gamma = x, y, z$. The function $\varphi_{\alpha\beta}^{(1)}(\tau)$ in (3) determines the dielectric tensor

$$\varepsilon_{\alpha\beta} = 1 + \int_0^{\infty} \varphi_{\alpha\beta}^{(1)}(\tau) e^{i\omega\tau} d\tau, \quad (5)$$

*rot = curl.

$$\varepsilon_{\alpha\beta}(\omega) = \varepsilon_{\alpha\beta}^*(-\omega). \quad (6)$$

In what follows, only transparent media will be considered. We can then write [4]

$$\varepsilon_{\alpha\beta}(\omega) = \varepsilon_{\beta\alpha}^*(\omega). \quad (7)$$

For simplicity, we shall not consider spatial dispersion, i.e., we shall assume that $\varphi_{\alpha\beta}^{(1)}(t - t_1)$ does not depend on $\mathbf{r} - \mathbf{r}_1$, or $\varepsilon_{\alpha\beta}(\omega)$ on $\mathbf{k}$. Even in this case, $\varphi_{\alpha\beta\gamma}^{(2)}(t - t_1)$ does not in general depend on $\mathbf{r} - \mathbf{r}_1, \mathbf{r}_1 - \mathbf{r}_2$ (for example, for a cold plasma). In what follows, however, the dependence of $\varphi_{\alpha\beta\gamma}^{(2)}$ on the coordinates is not expressed explicitly, inasmuch as it does not play a decisive role and complicates the formulas. All the results obtained are easily generalized to the case of spatial dispersion.

It must be kept in mind that $\varphi_{\alpha\beta\gamma}^{(2)}$ vanishes in certain cases, for example, for bodies possessing a center of symmetry (in the absence of the dependence of $\varphi_{\alpha\beta\gamma}^{(2)}$ on $\mathbf{r}, \mathbf{r}_1$, or $\mathbf{r}_2$). In such a case, it is necessary to take the terms of third order in the field into account— $\varphi_{\alpha\beta\gamma\delta}^{(3)}$. We apply the method set forth below in this case; however, we shall not consider it specially since the effects determined below are chiefly connected with the term $\varphi_{\alpha\beta\gamma}^{(2)}$, which is quadratic in the field.

3. We first write down the fundamental relations of the linear approximation in a form convenient for subsequent expansion. The solution of the linearized equations of Maxwell has the form

$$\begin{aligned} \mathbf{E} &= V^{-1/2} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} c_{\mathbf{p}} \mathbf{E}(\mathbf{p}) \exp \{i(\mathbf{p}\mathbf{r} - \omega_{\mathbf{p}}t)\}, \\ \mathbf{H} &= V^{-1/2} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} c_{\mathbf{p}} \mathbf{H}(\mathbf{p}) \exp \{i(\mathbf{p}\mathbf{r} - \omega_{\mathbf{p}}t)\}, \end{aligned} \quad (8)$$

where the $c_{\mathbf{p}}$ are constants. Here and in what follows, we shall denote the wave vector of the

harmonic with positive frequency by $\mathbf{k}$, and the wave vector of the harmonic with negative frequency by $\mathbf{k}_-$, so that

$$\mathbf{k}_- = -\mathbf{k}, \quad \omega_{\mathbf{k}_-} = -\omega_{\mathbf{k}}, \quad c_{\mathbf{k}_-} = c_{\mathbf{k}}^*,$$

$$\mathbf{E}(\mathbf{k}_-) = \mathbf{E}^*(\mathbf{k}), \quad \mathbf{H}(\mathbf{k}_-) = \mathbf{H}^*(\mathbf{k}). \quad (9)$$

$\mathbf{E}(\mathbf{p})$ and $\mathbf{H}(\mathbf{p})$ satisfy the equations ($\mathbf{p} = \mathbf{k}, \mathbf{k}_-$)

$$[\mathbf{pH}(\mathbf{p})]_{\alpha} = -(1/c) \varepsilon_{\alpha\beta} E_{\beta}(\mathbf{p}), \quad \omega_{\mathbf{p}} H(\mathbf{p}) = c [\mathbf{pE}(\mathbf{p})], \quad (10)$$

from which it follows that

$$[p_{\alpha} p_{\beta} - p^2 \delta_{\alpha\beta} + (\omega_{\mathbf{p}}^2/c^2) \varepsilon_{\alpha\beta}(\omega_{\mathbf{p}})] E_{\beta}(\mathbf{p}) = 0, \quad (11)$$

$$H_{\alpha}(\mathbf{k}) H_{\alpha}^*(\mathbf{k}) = \varepsilon_{\alpha\beta} E_{\alpha}^*(\mathbf{k}) E_{\beta}(\mathbf{k}). \quad (12)$$

Finally, to determine the meaning of $c_{\mathbf{p}}$ in (8), we must normalize the vectors $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$. For this purpose, we write down the expression for the electromagnetic density of the medium W (which has meaning only for a transparent medium [4]). By using the expansion (8) and the relation (12), we have (see [4])

$$\begin{aligned} W &= (16\pi)^{-1} \sum_{\mathbf{k}} \left[\frac{d(\omega \varepsilon_{\alpha\beta})}{d\omega} E_{\alpha}^*(\mathbf{k}) E_{\beta}(\mathbf{k}) + H_{\alpha}^*(\mathbf{k}) H_{\alpha}(\mathbf{k}) \right] c_{\mathbf{k}}^* c_{\mathbf{k}} \\ &= (16\pi)^{-1} \sum_{\mathbf{k}} \left[\frac{d(\omega \varepsilon_{\alpha\beta})}{d\omega} + \varepsilon_{\alpha\beta} \right] E_{\alpha}^*(\mathbf{k}) E_{\beta}(\mathbf{k}) c_{\mathbf{k}}^* c_{\mathbf{k}}, \end{aligned} \quad (13)$$

where the summation is carried out over all harmonics with positive frequencies. As in [4], we normalize $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$ so that W takes the form

$$W = \sum_{\mathbf{k}} c_{\mathbf{k}}^* c_{\mathbf{k}} \omega_{\mathbf{k}}, \quad (14)$$

i.e.,

$$(16\pi)^{-1} \left[\frac{d(\omega \varepsilon_{\alpha\beta})}{d\omega} + \varepsilon_{\alpha\beta} \right] E_{\alpha}^*(\mathbf{k}) E_{\beta}(\mathbf{k}) = \omega(\mathbf{k}). \quad (15)$$

In such a normalization, $|c_{\mathbf{k}}|^2 = c_{\mathbf{k}}^* c_{\mathbf{k}}$ has the dimensions of action, and the quantity $|c_{\mathbf{k}}|^2/\hbar$ must be interpreted as the number of quasiparticles with energy $\hbar\omega_{\mathbf{k}}$.

4. Let us now consider the nonlinear term $P^{(2)}$ in (2), which determines the interaction between the waves. Assuming this interaction to be sufficiently weak (amplitudes of $c_{\mathbf{k}}$ small), we shall seek the solution of the set (1)–(4) in the form

$$\begin{aligned} \mathbf{E} &= V^{-1/2} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} c_{\mathbf{p}}(t) [\mathbf{E}(\mathbf{p}) + \mathbf{E}'(\mathbf{p})] e^{i(\mathbf{p}\mathbf{r}-\omega_{\mathbf{p}}t)}, \\ \mathbf{H} &= V^{-1/2} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} c_{\mathbf{p}}(t) [\mathbf{H}(\mathbf{p}) + \mathbf{H}'(\mathbf{p})] e^{i(\mathbf{p}\mathbf{r}-\omega_{\mathbf{p}}t)}, \end{aligned} \quad (16)$$

where $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$ are the polarization vectors of the normal waves of linear approximation, determined by the relations (10) and (15), $c_{\mathbf{p}}(t)$ are

slowly varying amplitudes [in comparison with the exponents in (16)], $\mathbf{E}'(\mathbf{k})$ and $\mathbf{H}'(\mathbf{k})$ are small, slowly varying additions to the polarization vectors of the normal vibrations.

It will be clear from what follows that if $c_{\mathbf{p}}(t)$ is regarded as small of first order, then $\mathbf{E}'$ and $\mathbf{H}'$ will also be of first order smallness, and $dc_{\mathbf{p}}/dt$ will be of second order; the derivatives $d\mathbf{E}'/dt$, $d\mathbf{H}'/dt$ can also be regarded as small in comparison with $\mathbf{E}'$, $\mathbf{H}'$. Substituting (16) in (3), and limiting ourselves to terms up to second order smallness inclusively, we get

$$\begin{aligned} \frac{\partial P_{\alpha}^{(1)}}{\partial t} &= \frac{V^{-1/2}}{4\pi} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} e^{i(\mathbf{p}\mathbf{r}-\omega_{\mathbf{p}}t)} \{ i\omega_{\mathbf{p}} [\delta_{\alpha\beta} - \varepsilon_{\alpha\beta}(\omega_{\mathbf{p}})] c_{\mathbf{p}} E'_{\beta}(\mathbf{p}) \\ &+ E_{\beta}(\mathbf{p}) \int_0^{\infty} \varphi_{\alpha\beta}^{(1)}(\tau) e^{i\omega_{\mathbf{p}}\tau} [c_{\mathbf{p}}(t-\tau) - i\omega_{\mathbf{p}} c_{\mathbf{p}}(t-\tau)] d\tau \}. \end{aligned} \quad (17)$$

In the second term we expand $c_{\mathbf{p}}(t)$ in a Fourier integral

$$c_{\mathbf{p}}(t-\tau) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{c}_{\mathbf{p}}(\nu) e^{-i\nu(t-\tau)} d\nu. \quad (18)$$

Substituting this in (17) and carrying out the elementary transformations in the second term, we get

$$-iE_{\beta}(\mathbf{p}) \int_{-\infty}^{\infty} d\nu e^{-i\nu t} (\nu + \omega_{\mathbf{p}}) [\varepsilon_{\alpha\beta}(\nu + \omega_{\mathbf{p}}) - \delta_{\alpha\beta}] \tilde{c}_{\mathbf{p}}(\nu). \quad (19)$$

The smallness of the change in $c_{\mathbf{p}}(t)$ means that $\tilde{c}_{\mathbf{p}}(\nu)$ is substantially different from zero only for $\nu \ll \omega_{\mathbf{p}}$. We can therefore write in (19)

$$\begin{aligned} &(\nu + \omega_{\mathbf{p}}) [\varepsilon_{\alpha\beta}(\nu + \omega_{\mathbf{p}}) - 1] \\ &\approx \omega_{\mathbf{p}} [\varepsilon_{\alpha\beta}(\omega_{\mathbf{p}}) - \delta_{\alpha\beta}] + \nu \frac{d}{d\omega} [\omega(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})]_{\omega=\omega_{\mathbf{p}}}. \end{aligned} \quad (20)$$

Substituting (20) in (19), and carrying out the inverse summation of the Fourier components, we get

$$\begin{aligned} \frac{\partial P_{\alpha}^{(1)}}{\partial t} &= \frac{V^{-1/2}}{4\pi} \sum_{\mathbf{p}=\mathbf{k}, \mathbf{k}_-} e^{i(\mathbf{p}\mathbf{r}-\omega_{\mathbf{p}}t)} \\ &\times \left\{ i\omega_{\mathbf{p}} [\delta_{\alpha\beta} - \varepsilon_{\alpha\beta}(\omega_{\mathbf{p}})] [E_{\beta}(\mathbf{p}) + E'_{\beta}(\mathbf{p})] c_{\mathbf{p}} + \frac{dc_{\mathbf{p}}}{dt} E_{\beta}(\mathbf{p}) \right. \\ &\times \left. \frac{d}{d\omega} [\omega(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})]_{\omega=\omega_{\mathbf{p}}} \right\}. \end{aligned} \quad (21)$$

Substituting (16) in (4), we compute $\partial P_{\alpha}^{(2)}/\partial t$ (with accuracy to terms of second order):

$$\begin{aligned} \frac{\partial P_{\alpha}^{(2)}}{\partial t} &= -\frac{i}{4\pi V} \sum_{\mathbf{p}} e^{i\mathbf{p}\mathbf{r}} \sum_{\mathbf{p}'+\mathbf{p}''=\mathbf{p}} c_{\mathbf{p}'} c_{\mathbf{p}''} \sigma_{\alpha\beta\gamma}(\omega_{\mathbf{p}'}, \omega_{\mathbf{p}''}) \\ &\times E_{\beta}(\mathbf{p}') E_{\gamma}(\mathbf{p}'') \exp[i(\omega_{\mathbf{p}'} + \omega_{\mathbf{p}''})t], \end{aligned} \quad (22)$$

where ¹⁾

$$\begin{aligned} \sigma_{\alpha\beta\gamma}(\omega', \omega'') &= 1/2 (\omega' + \omega'') [\chi_{\alpha\beta\gamma}(\omega' + \omega'', \omega') \\ &+ \chi_{\alpha\gamma\beta}(\omega' + \omega'', \omega'')], \\ \chi_{\alpha\beta\gamma}(\omega', \omega'') &= \int_0^\infty d\tau' \int_0^\infty d\tau'' \varphi_{\alpha\beta\gamma}(\tau', \tau'') \exp [i(\omega'\tau' + \omega''\tau'')]. \end{aligned} \quad (23)$$

Substituting (21) and (22) in (1) and (2), we equate the identical Fourier $\mathbf{p}$ components, use (10), and then eliminate $\mathbf{H}'(\mathbf{p})$ from the resulting equations. We then get a set of equations for $E'_\alpha(\mathbf{p})$:

$$\begin{aligned} \frac{c}{\omega_p} c_p (\rho_\alpha \rho_\beta - \rho^2 \delta_{\alpha\beta} + \frac{\omega_p^2}{c^2} \varepsilon_{\alpha\beta}) E'_\beta \\ = -i \frac{dc_p}{dt} \left[\varepsilon_{\alpha\beta} + \frac{d(\omega_p \varepsilon_{\alpha\beta})}{d\omega_p} \right] E_\beta \\ - V^{-1/2} \sum_{\mathbf{p}'+\mathbf{p}''=\mathbf{p}} c_{p'} c_{p''} \sigma_{\alpha\beta\gamma}(\omega_{p'}, \omega_{p''}) E_\beta(\mathbf{p}') E_\gamma(\mathbf{p}'') \\ \times \exp [i(\omega_p - \omega_{p'} - \omega_{p''}) t]. \end{aligned} \quad (25)$$

This set differs from the set of equations (11) for $E'_\alpha(\mathbf{p})$ only in the presence of the right hand side. Consequently, its numerator is equal to zero. For the set of equations (25), it is necessary that the right side of this set be orthogonal to the solutions of the transposed set without the right-hand side, i.e., the vectors $\tilde{E}'_\alpha(\mathbf{p})$ satisfying the equation

$$[\rho_\alpha \rho_\beta - \rho^2 \delta_{\alpha\beta} + (\omega_p^2/c^2) \varepsilon_{\alpha\beta}(\omega_p)] \tilde{E}_\beta = 0. \quad (26)$$

On the basis of (7), one easily obtains

$$\tilde{E}_\alpha(\mathbf{p}) = E_\alpha^*(\mathbf{p}). \quad (27)$$

Thus the consistency condition (25) takes the form

$$\begin{aligned} i \frac{dc_p}{dt} \left\{ \varepsilon_{\alpha\beta}(\omega_p) + \frac{d}{d\omega_p} [\omega_p \varepsilon_{\alpha\beta}(\omega_p)] \right\} E_\alpha^*(\mathbf{p}) E_\beta(\mathbf{p}) \\ + V^{-1/2} \sum_{\mathbf{p}'+\mathbf{p}''=\mathbf{p}} c_{p'} c_{p''} \sigma_{\alpha\beta\gamma}(\omega_{p'}, \omega_{p''}) E_\alpha^*(\mathbf{p}) E_\beta(\mathbf{p}') E_\gamma(\mathbf{p}'') \\ \times \exp [i(\omega_p - \omega_{p'} - \omega_{p''}) t] = 0. \end{aligned} \quad (28)$$

Using the normalization condition (15), we see that the expression in the curly brackets in (28) is equal to $16\pi\omega_p$, and we get finally

$$i \frac{dc_p}{dt} = \sum_{\mathbf{p}', \mathbf{p}''} V_{\mathbf{p} \mathbf{p}' \mathbf{p}''} c_{p'} c_{p''} \exp [i(\omega_p - \omega_{p'} - \omega_{p''}) t], \quad (29)$$

where

$$\begin{aligned} V_{\mathbf{p} \mathbf{p}' \mathbf{p}''} &= -V^{-1/2} (16\pi\omega_p)^{-1} \sigma_{\alpha\beta\gamma}(\omega_{p'}, \omega_{p''}) E_\alpha^*(\mathbf{p}) E_\beta(\mathbf{p}') E_\gamma(\mathbf{p}'') \\ &\quad \text{for } \mathbf{p} = \mathbf{p}' + \mathbf{p}'', \quad (30) \\ V_{\mathbf{p} \mathbf{p}' \mathbf{p}''} &= 0 \quad \text{for } \mathbf{p} \neq \mathbf{p}' + \mathbf{p}''. \end{aligned} \quad (30a)$$

We have thus obtained an equation determining

¹⁾The analytic properties of the tensor $\sigma_{\alpha\beta\gamma}(\omega', \omega'')$ were studied by Kogan.^[5]

the change with time of the amplitudes of the interacting waves, which is identical in form with the dynamic equation for the waves obtained previously.^[1] For calculation of $V_{\mathbf{p} \mathbf{p}' \mathbf{p}''}$ in explicit form, it is necessary to know the function $\varphi_{\alpha\beta\gamma}^{(2)}(\tau_1, \tau_2)$, which determines the nonlinear response of the medium to the external electric field.

According to^[3], the following expressions are valid for these functions:

$$\varphi_{\alpha\beta\gamma}^{(2)}(t_1, t_2) = (i/\hbar)^2 \langle [Q_\alpha(t_1 + t_2), Q_\beta(t_2)]_-, Q_\gamma]_- \rangle_0. \quad (31)$$

However, for a number of concrete cases in which the quantum properties of the medium are inessential, in particular for a classical plasma, the quantity $\varphi_{\alpha\beta\gamma}^{(2)}(\tau_1, \tau_2)$ is easily computed directly by means of the kinetic or hydrodynamic equations, expressing the polarization vector of the medium in terms of the field intensity, with accuracy up to second order. Proceeding in this fashion in the case of a "cold" plasma in a strong magnetic field ($H^2/8\pi \gg nT$) we get the equations for $V_{\mathbf{p} \mathbf{p}' \mathbf{p}''}$ found for this same case in^[1] by a somewhat different method.

The "matrix elements" $V_{\mathbf{p} \mathbf{p}' \mathbf{p}''}$ satisfy some general symmetry relations. Two of them are easily obtained immediately:

$$V_{\mathbf{p} \mathbf{p}' \mathbf{p}''} = V_{\mathbf{p} \mathbf{p}'' \mathbf{p}'}, \quad (32)$$

$$V_{\mathbf{p} \mathbf{p}' \mathbf{p}''} = -V_{\mathbf{p}_- \mathbf{p}'_- \mathbf{p}''_-}. \quad (33)$$

The first follows from (30) and (23), and the second is obtained if we take the complex conjugate of (30) and consider that $c_{\mathbf{p}}^* = c_{\mathbf{p}} [see (9)]$.

In contrast with (33) two other general relations are satisfied only for

$$\omega_p = \omega_{p'} + \omega_{p''}, \quad \mathbf{p} = \mathbf{k}, \mathbf{k}_- \quad (34)$$

and have the form

$$V_{\mathbf{k} \mathbf{k}' \mathbf{k}''} = V_{\mathbf{k}'' \mathbf{k}' \mathbf{k}}, \quad (35a)$$

$$V_{\mathbf{k} \mathbf{k}' \mathbf{k}_-} = -V_{\mathbf{k}_- \mathbf{k}' \mathbf{k}}. \quad (35b)$$

The relations (35) were obtained in^[1] for the special case considered there on the basis of a concrete form of the matrix elements of $V_{\mathbf{k} \mathbf{k}' \mathbf{k}''}$. We shall not give them here for the general case since this proof is rather cumbersome and the relation (35) is not necessary in this work. We only note that one can get (35a) by starting out from the correspondence between the quantum and classical theories of the interaction of electromagnetic waves in the medium.²⁾ Then (35a) follows from

²⁾We note that the effects of the interaction of the waves appear at such high intensities (the number of photons is $n_{\mathbf{k}} = c_{\mathbf{k}}^* c_{\mathbf{k}} / \hbar \gg 1$) that the quantum corrections are, as a rule, unimportant. Moreover, the quantum consideration, which is

the skew symmetry of the matrix element of the S-matrix [see, for example, ^[6]] which describes the decay of a photon traveling in a transparent medium with energy $\hbar\omega_k$ and momentum $\hbar k$ into two waves: $(\omega_{k_1}, k_1); (\omega_{k_2}, k_2)$, such that (34) is satisfied and $k = k_1 + k_2$. So far as (35b) is concerned, it is simply obtained from (35a) and (33).

By means of the dynamic equation (29) and the relation (35) one can study the decay instability of waves which was considered previously by Oraevskii and Sagdeev, ^[7] and can also obtain the kinetic equation describing the interaction of a large number of waves with randomly distributed phases. All this was done in ^[1] for a plasma; however, inasmuch as a concrete form is not used here for the matrix elements, all these results are valid in an arbitrary transparent medium. ³⁾

5. As an example of the application of the dynamic equation (29), let us consider the effect of frequency doubling in a transparent medium. A report on the observation of this effect in crystalline quartz is given in ^[2].

Let the wave $[k, \omega(k)]$ be propagated in the medium. Then, on the basis of the dynamic equation (29), a wave appears with the double wave vector $k_1 = 2k$; its amplitude satisfies the equation

$$i \frac{dc_{k_1}}{dt} = V_{k, k k} c_k^2 e^{i(\omega_1 - 2\omega)t}, \quad (36)$$

$$\omega_1 = \omega(k_1) = \omega(2k), \quad \omega = \omega(k).$$

We then get (neglecting the time dependence of the initial amplitude, which can be done when c_k is large in comparison with c_{k_1})

$$c_{k_1}(t) = (2\omega - \omega_1)^{-1} V_{k, k k} c_k^2 e^{i(\omega_1 - 2\omega)t}. \quad (37)$$

The equation of the wave with the wave vector $k_1 = 2k$ is the following on the basis of expansion of (8),

$$\mathcal{E}_1(r, t) = V^{-1/2} E(k_1) c_{k_1} e^{i(k_1 r - \omega_1 t)}$$

$$= V^{-1/2} E(k_1) (2\omega - \omega_1)^{-1} V_{k, k k} c_k^2 e^{i(k_1 r - 2\omega t)}. \quad (38)$$

Thus it follows that the second harmonic is twice the frequency in comparison with the initial wave. ⁴⁾

useful in obtaining general relations of the type (35) does not lead in sufficiently simple fashion to the explicit expressions (30) for the matrix elements. We shall therefore not consider it here.

³⁾A report was made in ^[8,9] on the experimental observation in crystals of a phenomenon that is the opposite of wave decay, namely, the harmonic $(\omega_1 + \omega_2, k_1 + k_2)$ appears in the superposition of two waves $(\omega_1, k_1), (\omega_2, k_2)$. We note further that one of the decay cases was discussed previously by Akhmanov and Khokhlov^[10] as a possible mechanism of wave amplification.

⁴⁾Inasmuch as $2\omega \neq \omega_1 = \omega(2k)$ as a consequence of the nonlinearity of the dispersion law, then such a wave must be considered as "forced."

Equation (38) is valid for

$$|V_{k, k k} c_k / (2\omega - \omega_1)| \ll 1, \quad |2\omega - \omega_1| \ll \omega_1. \quad (39)$$

The first of the inequalities (39) is the condition of the smallness of c_{k_1} in comparison with c_k , while the second is the condition of the slowness of change of $c_{k_1}(t)$. Both conditions can be satisfied simultaneously for

$$|V_{k, k k} c_k / \omega_1| \ll |(2\omega - \omega_1) / \omega_1| \ll 1. \quad (40)$$

If some different waves with the same initial frequency ω are propagated in the medium, then they also can lead to the appearance of waves with double frequency (such a case is encountered below). The amplitudes of the latter will be determined by an equation similar to (37):

$$c_{k_1} = (2\omega - \omega_1)^{-1} \sum_{k_1 = k' + k''} V_{k_1, k' k''} c_{k'} c_{k''} e^{i(\omega_1 - 2\omega)t}, \quad (41)$$

where k' and k'' are the wave vectors of the initial wave.

We now consider in detail the form of the matrix element $V_{k_1, k' k''}$. It follows from (30) that one can describe it in the following fashion:

$$V_{k_1, k' k''} = -V^{-1/2} (\omega / 8\pi\omega_1) E^*(k_1) u(k', k''), \quad (42)$$

where the vector $u(k', k'')$ is determined by the equation

$$u_\alpha(k', k'') = (2\omega)^{-1} \epsilon_{\alpha\beta\gamma}(\omega, \omega) E_\beta(k') E_\gamma(k''). \quad (43)$$

We now apply these considerations to crystalline quartz which was investigated in ^[2]. The latter is a uniaxial crystal with symmetry class D_3 , containing a three-fold symmetry axis (the z axis) and three two-fold symmetry axes in the xy plane; one of these axes is assumed directed along x . It then follows that the tensor $\sigma_{\alpha\beta\gamma}(\omega)$ will have the form (see ^[4], page 106)

$$\sigma_{x, xx} = -\sigma_{x, yy} = -\sigma_{y, xy} = 2\omega a, \quad (44)$$

$$\sigma_{x, yz} = -\sigma_{y, xz} = 2\omega b,$$

where a and b are certain functions of the frequency ω .

Substituting (44) in (43), we get the vector u in the form

$$u_x = a [E_x(k') E_x(k'') - E_y(k') E_y(k'')]$$

$$+ b [E_y(k') E_z(k'') + E_y(k'') E_z(k')],$$

$$u_y = -b [E_x(k') E_z(k'') + E_x(k'') E_z(k')]$$

$$- a [E_x(k') E_y(k'') + E_x(k'') E_y(k')], \quad (45)$$

$$u_z = 0.$$

The tensor $\epsilon_{\alpha\beta}$ is diagonal in the chosen set of coordinates, whence

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon, \quad \epsilon_{zz} = \epsilon'. \quad (46)$$

It follows from the absence of absorption that ϵ, ϵ' , and the polarization vectors $\mathbf{E}(\mathbf{k})$ are real.

We now consider three cases separately.

A. The incident ray is directed along the x axis (that is, along one of the two-fold axes). For arbitrary polarization, it yields ordinary and extraordinary waves having the same frequency and direction of propagation, but different values of the wave vectors and different polarizations; the ordinary wave is polarized along the y axis, and the extraordinary wave along the z axis. Consequently, $E_x(\mathbf{k}') = E_x(\mathbf{k}'')$, and the vector $\mathbf{u}$ has only an x component. Inasmuch as a secondary wave is also propagated along the x axis, $E_x(\mathbf{k}_1) = 0$ and consequently $\mathbf{E}(\mathbf{k}_1) \cdot \mathbf{u}(\mathbf{k}', \mathbf{k}'') = 0$. Therefore $V_{\mathbf{k}_1 \mathbf{k}' \mathbf{k}''} = 0$. This means that the wave with double frequency cannot arise in this case.

B. The incident ray is directed along the y axis. For arbitrary polarization, it again yields an ordinary and an extraordinary wave. The first is polarized along the x axis, and the second along the z axis. For both waves, $E_y(\mathbf{k}) = 0$. For the secondary wave, $E_y(\mathbf{k}_1) = 0$ also. Consequently,

$$\mathbf{E}(\mathbf{k}_1) \mathbf{u}(\mathbf{k}', \mathbf{k}'') = E_x(\mathbf{k}_1) u_x(\mathbf{k}', \mathbf{k}''). \quad (46a)$$

It follows from (45) that $u_x(\mathbf{k}', \mathbf{k}'') \neq 0$ only in the ordinary wave. Consequently, in this case, only the ordinary wave produces a harmonic with double frequency, and the latter is also polarized along x .

The intensity of the wave with double frequency is determined by the energy flux

$$S = (c/4\pi) [\mathbf{E} \mathbf{H}] = (c/4\pi) n E^2 \quad (46b)$$

(the latter is written with account of the transverse character; the field intensity here is assumed to be real). The vector $\mathbf{n}$ is equal in magnitude to the index of refraction of the wave (here, the ordinary wave) and is directed along the wave vector.

The average value of the energy flux, expressed in terms of the complex intensity vector $\mathcal{E}$, has the form

$$S_1 = (cn_1/8\pi) \mathcal{E}_1^* \mathcal{E}_1, \quad (47)$$

where the index 1 denotes that the quantities refer to the secondary wave. We substitute $\mathcal{E}_1$ here from (38) and the expression for the matrix element from (42), (45):

$$|V_{\mathbf{k}, \mathbf{k} \mathbf{k}}|^2 = V^{-1} (\omega^2 / (8\pi)^2 \omega_1^2) a^2 E_x^2(\mathbf{k}_1) E_x^4(\mathbf{k}). \quad (48)$$

The values of the polarization vectors here are determined by the normalization condition (15), from which it follows that

$$E^2(\mathbf{k}) = 16\pi\omega(\omega d\epsilon/d\omega + 2\epsilon)^{-1} \approx 8\pi\omega/n^2, \quad (49)$$

$$E^2(\mathbf{k}_1) \approx 8\pi\omega_1/n_1^2,$$

where n and n_1 are the indices of refraction of the ordinary wave, corresponding to the first and second harmonics.

Substituting (49), (48), and (38) in (47), we finally obtain

$$S_1 = \frac{2\pi}{c} \frac{a^2}{n_1 n^2 (n_1 - n)^2} S_0^2 = \frac{2\pi}{c} \frac{a^2}{n_1 n^2 (n_1 - n)^2} S^2 \cos^4 \varphi, \quad (50)$$

where n and n_1 are the indices of refraction of the ordinary ray, corresponding to the wavelength λ (initial) and $\lambda/2$; S_0 is the energy flux of the initial ordinary wave, and S is the total energy flux or the initial ray in the crystal, φ is the angle between the direction of the polarization of the incident ray (at incidence on the crystal) and the x axis. Thus the intensity of the second harmonic here depends significantly on the polarization of the incident ray.

C. The incident wave is directed along the z axis, that is, the principal axis of the crystal; $E_z(\mathbf{k}) = 0$.

In this case the difference between the ordinary and extraordinary waves vanishes; the primary wave can have arbitrary polarization in the xy plane, i.e., degeneracy sets in. In this connection, it is necessary to make some observations relative to the initial equation (36). It was obtained from (25) by starting out from the fact that the right-hand side of (25) should be orthogonal to the solutions of the transposed system without the right-hand side. But, owing to the degeneracy, the homogeneous system now satisfies not a single vector as was assumed earlier, but all vectors lying in the xy plane. Therefore, the right-hand side of (25) must be orthogonal to any vector lying in the xy plane and, consequently, the x and y components for it must vanish.

Substituting $\mathbf{p} = \mathbf{k}_1$, $\mathbf{p}' = \mathbf{p}'' = \mathbf{k}$ and $\epsilon_{\alpha\beta}$ in (25) from (46), we get the right side of (25) in the form

$$\frac{d\epsilon_{\mathbf{k}_1}}{dt} \left(\epsilon + \frac{d(\omega\epsilon)}{d\omega} \right) \mathbf{E}(\mathbf{k}_1) - 2\omega V^{-1/2} c_{\mathbf{k}}^2 \mathbf{u}(\mathbf{k}, \mathbf{k}) \exp[i(\omega_{\mathbf{k}_1} - 2\omega_{\mathbf{k}})t] = 0. \quad (51)$$

Here we have introduced the vector $\mathbf{u}$ in accord with (43), and taken into account the fact that $E_z(\mathbf{k}_1) = 0$ ($\mathbf{k}_1 \parallel \text{Oz}$) and $u_z(\mathbf{k}, \mathbf{k}) = 0$ [on the basis of (45)].

From Eq. (51) follows, in the first place, the previous Eq. (36), and in the second place the fact that $\mathbf{E}(\mathbf{k}_1)$ and $\mathbf{u}(\mathbf{k}, \mathbf{k})$ are parallel. Thus the polarization of the wave with double frequency is identical with the polarization vector $\mathbf{u}$ with components

$$u_x = a[E_x^2(\mathbf{k}) - E_y^2(\mathbf{k})], \quad u_y = -2aE_x(\mathbf{k})E_y(\mathbf{k}), \quad u_z = 0, \quad (52)$$

where $E_x(\mathbf{k})$, $E_y(\mathbf{k})$ are the components of the polarization vector of the initial wave. It then follows that if $E(\mathbf{k})$ makes the angle φ with the x axis, while the electric field of the secondary wave makes the angle φ_1 , then

$$\varphi_1 = 2(\pi - \varphi). \quad (53)$$

Let us compute the intensity of the second harmonic in this case. By taking it into account that $E(\mathbf{k}_1)$ and $u(\mathbf{k}, \mathbf{k})$ in the matrix element (42) are parallel, we get

$$\begin{aligned} |V_{\mathbf{k}, \mathbf{k}'}|^2 &= V^{-1} (\omega/8\pi\omega_1)^2 E^2(\mathbf{k}_1) u^2(\mathbf{k}, \mathbf{k}) \\ &= V^{-1} (a\omega/8\pi\omega_1)^2 E^2(\mathbf{k}_1) E^4(\mathbf{k}). \end{aligned} \quad (54)$$

By substituting (54), (49), and (38) in (47), we get

$$S_1 = \frac{2\pi}{c} \frac{a^2}{n_1 n^2 (n_1 - n)^2} S^2, \quad (55)$$

where S_1 and S are the energy fluxes of the secondary and initial wave, and n_1 and n are the corresponding indices of refraction. In this case, the intensity of the wave of double frequency does not depend on the polarization of the incident beam.

We note that many qualitative features of the dependence of the intensity of the secondary wave on the polarization of the incident wave pointed out above are identical with the results of the analysis carried out in [2] by another method, and are confirmed by observation. The formula given in [2] for the intensity is in error.

From (55), one can make a numerical estimate of the effect under consideration. In fact, the quantity a^{-1} has the dimensions of the field intensity and must be equal in order of magnitude to the value of the atomic field. Substituting in (55) the

expression $S \sim (c/4\pi) E^2$, where E is the field intensity in the medium and $n_1 - n \sim n^{-2}$, we get

$$S_1/S \sim 10^4 (aE)^2. \quad (55a)$$

For $a \sim 10^{-9}$ cm/V, $E \sim 10^5$ V/cm (a feasible value) we get $S_1/S \sim 0.01$ per cent.

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REGGE POLES IN QUANTUM FIELD THEORY

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The condition of unitarity and the dispersion relations are used to obtain the asymptotic behavior of the meson-meson scattering amplitude in the region of large momentum transfer.

1. INTRODUCTION

RECENTLY a new method has been introduced in quantum field theory which enables one to investigate bound states and resonances [1-4]. This method is based on the study of the asymptotic behavior of scattering amplitudes in the region of large momentum transfer.

From the point of view of the new method the basic objects of investigation in quantum field theory must be the Regge pole trajectories. The explicit calculation of Regge trajectories presents considerable difficulties and requires the use of approximation methods. At present there exist two approaches: 1) solution of the Bethe-Salpeter equations [5,6] and 2) summation of diagrams by the method of the renormalization group [7]. In this paper an approximation method is presented based on unitarity and dispersion relations.

2. THE CONDITION OF UNITARITY IN THE STEPWISE APPROXIMATION

We restrict ourselves to a discussion of the theory of neutral scalar mesons with contact interaction $\lambda\phi^3$. By applying Cutkosky's rules [8] to the diagrams shown in Fig. 1 we write the condition of unitarity ¹⁾:

$$A_{13}(sz) = \frac{1}{16\pi^2} \sqrt{\frac{q^2}{q^2 + m^2}} \int dz_1 \times \int dz_2 \frac{1}{\sqrt{k(z_1 z_2)}} A_3^{II}(sz_2) A_3(s z_1);$$

$$A(s z) = A(s + i\epsilon, z), \quad A^{II}(s z) = A(s - i\epsilon, z),$$

$$k(z_1 z_2) = z^2 + z_1^2 + z_2^2 - 1 - 2z z_1 z_2, \quad s = 4(q^2 + m^2), \\ t = 2q^2(z - 1), \quad t_1 = 2q^2(z_1 - 1), \quad t_2 = 2q^2(z_2 - 1), \quad (1)$$

¹⁾In this paper we use an approximation which does not take into account the contribution to the scattering amplitude of the singularities of the invariant u .

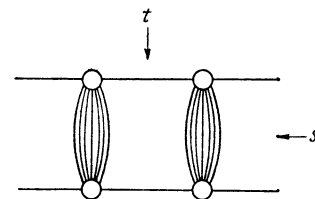


FIG. 1

where s is the square of the total energy, $-t$ is the square of the transferred momentum, m is the meson mass. Integration in (1) is carried out over the region

$$z > z_1 z_2 + \sqrt{(z_1^2 - 1)(z_2^2 - 1)}.$$

On substituting (1) into the dispersion relation

$$A_1(s z) = \frac{1}{\pi} \int dz' \frac{A_{13}(s z')}{z' - z}$$

and integrating over z' we obtain

$$A_1(s z) = -\frac{1}{32\pi^3} \sqrt{\frac{q^2}{q^2 + m^2}} \int dz_1 \int dz_2 \frac{1}{\sqrt{k(z_1 z_2)}} \times \ln \frac{z - z_1 z_2 + \sqrt{k(z_1 z_2)}}{z - z_1 z_2 - \sqrt{k(z_1 z_2)}} A_3^{II}(s z_2) A_3(s z_1). \quad (2)$$

Noting that

$$A_1(s z) = -\frac{1}{2} i [A(s z) - A^{II}(s z)],$$

we rewrite (2) in the form

$$A(s z) = A^{II}(s z) - \frac{i}{16\pi^3} \sqrt{\frac{q^2}{q^2 + m^2}} \int dz_1 \int dz_2 \frac{1}{\sqrt{k(z_1 z_2)}} \times \ln \frac{z - z_1 z_2 + \sqrt{k(z_1 z_2)}}{z - z_1 z_2 - \sqrt{k(z_1 z_2)}} A_3^{II}(s z_2) A_3(s z_1).$$

We make an approximation analogous to that which is made in obtaining equations of the Bethe-Salpeter type [9], viz., we set

$$A^{II}(s z) = -\frac{\lambda^2}{2q^2(z - 1) - m^2},$$

$$A_3^{II}(s z_2) = \pi \lambda^2 \delta [2q^2(z_2 - 1) - m^2].$$

We then obtain

$$\begin{aligned}
 A(s) = & -\frac{\lambda^2}{2q^2(z-1)-m^2} \\
 & -\frac{i\lambda^2}{32\pi^2\sqrt{q^2(q^2+m^2)}} \int dz_1 \frac{1}{\sqrt{k(z z_1 z_2)}} \\
 & \times \ln \frac{z-z_1 z_2 + \sqrt{k(z z_1 z_2)}}{z-z_1 z_2 - \sqrt{k(z z_1 z_2)}} A_3(s z_1); \\
 z_2 = & 1 + m^2/2q^2.
 \end{aligned} \quad (3)$$

Equation (3) is the unitarity condition in the step-wise approximation.

3. ASYMPTOTIC BEHAVIOR FOR LARGE MOMENTUM TRANSFER

As $t \rightarrow \infty$ ($z \rightarrow \infty$) Eq. (3) assumes the following form

$$A(st) = -\frac{\lambda^2}{t} - \frac{i\lambda^2}{8\pi^2\sqrt{s(s-4m^2)}} \int dt_1 \frac{1}{t} \ln(t_1-t) A_3(st_1). \quad (4)$$

We multiply (4) by t and differentiate with respect to t . Then, taking into account the dispersion relation

$$A(st) = \frac{1}{\pi} \int dt_1 \frac{A_3(st_1)}{t_1-t},$$

we obtain the simple differential equation

$$\begin{aligned}
 t \frac{d}{dt} A(st) &= \alpha(s) A(st); \\
 \alpha(s) &= -1 + i\lambda^2/8\pi\sqrt{s(s-4m^2)}.
 \end{aligned} \quad (5)$$

Equation (5) has the solution

$$A(st) = \beta(s) t^{\alpha(s)}.$$

The dependence of $\text{Re } \alpha(s)$ on s (the Regge trajectory) for $m=1$ and $\lambda^2=25$ is shown²⁾ in

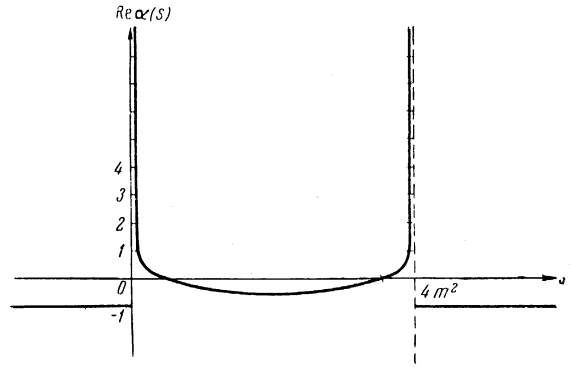


FIG. 2

Fig. 2. It can be seen from Fig. 2 that there exists a family of bound states [$d\text{Re } \alpha(s)/ds > 0$ for $2m^2 < s < 4m^2$]; there are no resonances and no "ghosts."

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CONSERVATION OF THE VECTOR CURRENT IN WEAK INTERACTIONS AND THE MASS DIFFERENCE IN ISOTOPIC MULTIPLETS. THE $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ DECAY

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The validity of the vector current conservation hypothesis in weak interactions is limited similarly to that of the isotopic invariance and hence, generally speaking, one should expect that in the decays of strongly interacting particles the vector coupling constant (which is linear in the particle-mass difference in isotopic multiplets) should be renormalized. In the first part of the paper a proof is given of the theorem (which, in strong interactions, is valid in all orders) that there is no such renormalization for the neutron and Σ hyperon β decay, and for the $\pi^\pm \rightarrow \pi^0 + e^\pm + \nu$ decay. The second part of the paper is devoted to a calculation of the probability of the $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay with radiative corrections taken into account. The use of the above theorem narrows greatly the limits of the theoretically predicted value of the probability.

1. VECTOR CURRENT CONSERVATION AND THE MASS DIFFERENCE

1. The vector current conservation hypothesis^[1,2] which was evoked to explain the lack of renormalization of the vector coupling constant in strong interactions has a strictly defined sense only when the mass difference in isotopic multiplets is neglected. The quantity $\delta j_\mu^{(i)}/\delta x_\mu$, where

$$j_\mu^{(i)} = \frac{1}{2} \bar{\Psi} \tau_i \gamma_\mu \Psi - \frac{i}{2} \left(\frac{\partial \varphi^*}{\partial x_\mu} T_i \varphi - \varphi^* T_i \frac{\partial \varphi}{\partial x_\mu} \right) + \text{terms corresponding to other strongly interacting particles and leptons} \quad (1)$$

(here Ψ and φ are the Heisenberg operators of the nucleon and π -meson fields) does not vanish if we take the terms linear in the mass difference into account.

We shall assume that in this case, too, the weak interaction can be written as $G j_\mu^+ j^\mu$; we would then in general expect a linear (in the mass differences) renormalization of the constant G in the decays of strongly interacting particles. It will, however, be shown in what follows that at least in the β decay of the neutron and of the Σ hyperon, and also in the $\pi^\pm \rightarrow \pi^0 + e^\pm + \nu$ decay, the violation of the isotopic invariance and the related renormalization of G are effects quadratic in the mass difference.

2. We assume that the total Hamiltonian H of

the field system can be written as the sum

$$H = H_1 + H_w + V + H_{el}, \quad (2)$$

where H_1 is the isotopically invariant part which includes the free Hamiltonian and all strong interactions, H_w is the weak interaction, V is the term containing the operators of the mass shift in isotopic multiplets (which, acting on the functions of physical states, produce the experimentally observed mass differences).

Two points of view are possible concerning the nature of this term. The accepted one is that the mass difference is due to electromagnetic interactions. From this point of view V represents the mass effect of electromagnetic interactions, in the form of a counterterm which permits the mass renormalization. However, by taking V into account, we do not in fact take into account all effects of electromagnetic interactions.

The remaining term H'_{el} describes the interaction of particles with a physical mass, and leads only to vertex effects and to the renormalization of wave functions. We can assume that these effects will be correctly taken into account by calculating the radiative corrections, and in the first part of the paper we do not consider them at all.

Since such an approach is not fully justified, we propose in the following another interpretation. Since the radiative corrections are very small ($\sim 1-2\%$), then the degree to which the isotopic invariance is violated in the decay of strongly interacting particles can be determined from the mass

differences in isotopic multiplets (which are very large in mesic and hyperonic multiplets). Disregarding the problem of the origin of the mass difference we shall take it into account phenomenologically [(term V in Eq. (1))]. Our thesis is that there is no linear effect in V.

We take the interaction in the form

$$V = \frac{1}{2} \delta m_{n,p} \int \bar{\Psi} (\tau_\beta + c_N) \Psi d^3x + \delta \mu_{\pi^0, \pi^\pm}^2 \int \varphi^* (T_3^2 + c_\pi) \varphi d^3x \quad \text{terms corresponding to mass differences of other isotopic multiplets} \quad (3)$$

where $\delta m_{n,p} = m_n - m_p$, $\delta \mu_{\pi^0, \pi^\pm}^2 = \mu_+^2 - \mu_0^2$, c_N and c_π are arbitrary constants related to the indeterminate values of the unperturbed multiplet masses (these constants determine the center-of-mass shift of the multiplets). The crucial assumption is that the perturbation theory can be applied to the interaction V.

3. Let us consider the $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay. The matrix element of this process is

$$M = (2\pi i) \delta(E^+ - E^0 - q_0) G l_\mu \int \langle \pi^0 | j_\mu^{(-)}(\mathbf{x}, 0) | \pi^+ \rangle e^{-iqx} d^3x, \quad (4)$$

$$\int \langle \pi^0 | j_\mu^{(-)}(\mathbf{x}, 0) | \pi^+ \rangle e^{-iqx} d^3x = \frac{\delta_{p^+, p^0+q}}{\sqrt{4E^+E^0}} [f(q^2)(p^+ + p^0)_\mu + h(q^2)q_\mu], \quad (5)$$

where $p^+ = (E^+, \mathbf{p}^+)$ and $p^0 = (E^0, \mathbf{p}^0)$ are the momenta of the π^+ and π^0 mesons, $q = (q_0, \mathbf{q})$ is the momentum transferred to leptons, $l_\mu = \bar{u}_\nu \gamma_\mu (1 + \gamma_5) v_e$ is the leptonic part of the current, and $j_\mu^{(-)} = 2^{-1/2} (j_\mu^{(1)} - i j_\mu^{(2)})$.

If the current $j_\mu^{(1)}$ is conserved, then $f(0) = 1$ and $h(q^2) \equiv 0$. In general $h(q^2) \sim \delta m$, where δm is the generalized symbol for the mass differences in Eq. (3).¹⁾

The time-like component of Eq. (5) for $q = 0$, at the point where $\delta_{p^+, p^0+q} = 1$, in the rest-mass system of the π^+ meson is:

$$\int \langle \pi^0 | j_0^{(-)}(\mathbf{x}, 0) | \pi^+ \rangle d^3x = \frac{\mu_+ + \mu_0}{\sqrt{4\mu_+\mu_0}} [f(0) + h(0) \frac{\mu_+ - \mu_0}{\mu_+ + \mu_0}] = f(0). \quad (6)$$

Eq. (6) is correct to terms quadratic in the mass differences.

¹⁾It should be noted that the term with $h(q^2)$ in Eq. (5) is not important in the expression for the decay probability since in addition to being small ($\approx \delta m$) as compared to the first term in Eq. (5), it contains the additional parameter m_e/μ_+ (where m_e is the electron mass) which is due to the action of q_μ on the leptonic bracket.

Let us introduce the notation

$$|\pi^+\rangle = \Psi^{(+)}(0) \equiv \Psi^{(+)},$$

$$|\pi^0\rangle = \Psi^{(0)}(0) \equiv \Psi^{(0)}, \quad j_\mu^{(i)}(\mathbf{x}, 0) = J_\mu^{(i)}(\mathbf{x}),$$

where $J_\mu^{(i)}(\mathbf{x})$ is the current operator in the Schroedinger representation, and $\Psi^{(+)}(t)$ and $\Psi^{(0)}(t)$ are the stationary single-meson states of the Hamiltonian H_1 . It should be noted that

$$j_\mu^{(i)}(\mathbf{x}, t) = e^{i(H_1+V)t} J_\mu^{(i)}(\mathbf{x}) e^{-i(H_1+V)t}.$$

The quantity $e^{iH_1t} J_\mu^{(1)}(\mathbf{x}) e^{-iH_1t}$ which represents the current (in the V-interaction representation) is at the same time a Heisenberg operator with respect to the isotopically invariant interactions included in H_1 . We have therefore

$$\frac{\partial}{\partial x_\mu} [e^{iH_1t} J_\mu^{(i)}(\mathbf{x}) e^{-iH_1t}] = 0.$$

In terms of the current $J_\mu^{(1)}(\mathbf{x})$, this relation becomes

$$\text{div } \mathbf{J}^{(i)}(\mathbf{x}) = i [H_1, J_0^{(i)}(\mathbf{x})]. \quad (7)$$

Integrating over the whole space we obtain the basic relation

$$[H_1, t^{(-)}] = 0, \quad (8)$$

where $t^{(-)} = \int J_0^{(-)}(\mathbf{x}) d^3x$. Finally we obtain:

$$f(0) = (\Psi^{(0)}, t^{(-)} \Psi^{(+)}), \quad (9)$$

where the whole dependence on the mass differences (or the interaction V) is contained in $\Psi^{(0)}$ and $\Psi^{(+)}$. It is evident that

$$(H_1 + V) \Psi^{(+);(0)} = \mu_{+;0} \Psi^{(+);(0)}. \quad (10)$$

Let us consider the complete set of states Φ_n of the unperturbed Hamiltonian H_1 . We have then the general formula of the perturbation theory for stationary states:

$$\Psi^{(+)} = \Phi^{(1)} + \sum_n' \frac{(\Phi_n, V \Phi^{(1)})}{E_n - \mu} \Phi_n, \quad (11)$$

$$\Psi^{(0)} = \Phi^{(0)} + \sum_n' \frac{(\Phi_n, V \Phi^{(0)})}{E_n - \mu} \Phi_n, \quad (11')$$

where $\Phi^{(1)}$ and $\Phi^{(0)}$ are the π^+ and π^0 meson states (for $\mathbf{p}^{+;0} = 0$) of the Hamiltonian H_1 . (The indices ⁽¹⁾ and ⁽⁰⁾ characterize the z component of the isotopic spin.) The summation $\sum_n'$ in Eq.

(11) is carried out over the complete set Φ_n excluding the π^+ -meson state $\Phi^{(1)}$. Correspondingly, the summation in Eq. (11') does not include the π^0 -meson state $\Phi^{(0)}$.

Since the interaction V is diagonal in the charge states, then the π^0 and π^- mesons are also absent

in Eq. (11); accordingly, there is no π^- or π^+ among the n states in Eq. (11'). Consequently, single-meson states are absent in Eqs. (11) and (11') among the n states, and the denominators in these formulas never vanish. (We assume that in the spectrum of the operator H_1 there are no other states with $p_n^2 = \mu^2$ apart of the single-meson states.)

Furthermore, using Eq. (9), we obtain

$$f(0) = (\Phi^{(0)}, t^{(-)} \Phi^{(1)}) + \sum_n \frac{1}{E_n - \mu} \{ (\Phi_n, V \Phi^{(1)}) (\Phi^{(0)}, t^{(-)} \Phi_n) + (\Phi_n, V \Phi^{(0)})^* (\Phi_n, t^{(-)} \Phi^{(1)}) \}. \quad (12)$$

Since $t^{(-)}$ is an integral of motion [cf. Eq. (8)], and since it obviously commutes with the total momentum of the field, it is necessary that $p_n^2 = \mu^2$. We found, however, that there are no such states among the n states in Eq. (12) and, therefore, there is no linear effect in V , i.e.,

$$f(0) = (\Phi^{(0)}, t^{(-)} \Phi^{(1)}). \quad (13)$$

It can be easily seen that $t^{(i)}$ are infinitesimal rotation operators in the isotopic spin space. Since $t^{(-)} = 2^{-1/2}(t^{(1)} - it^{(2)})$, then the matrix element $(\Phi^{(0)}, t^{(-)} \Phi^{(1)})$ in Eq. (13) is equal to unity²⁾, thus proving the theorem that the constant G is not renormalizable in the approximation linear in the mass differences.

4. The proof of the theorem is carried out analogously for the β decay ($n \rightarrow p + e + \nu$) and for the β decay of Σ hyperons. Thus, e.g., the β decay vertex operator Γ_μ should be written in the form

$$\Gamma_\mu = a(q^2) \gamma_\mu + b(q^2) q_\mu + c(q^2) \sigma_{\mu\nu} q^\nu. \quad (14)$$

The contribution of the form factors b and c is negligibly small, and for the quantity $a(0)$ we find easily a relation of the type (6):

$$a(0) = \sqrt{2} \int \langle p | j_0^{(+)}(x, 0) | n \rangle d^3x,$$

$$j^{(+)} = 2^{-1/2}(j^{(1)} + ij^{(2)}). \quad (15)$$

The consequent calculation is carried out in Sec. 3. As a result, we obtain an equation analogous to Eq. (13)

$$a(0) = \sqrt{2} (\Phi^{(1/2)}, t^{(+)} \Phi^{(-1/2)}), \quad (16)$$

where $\Phi^{(-1/2)}$ and $\Phi^{(1/2)}$ are the neutron and proton states of the Hamiltonian H_1 . Repeating the

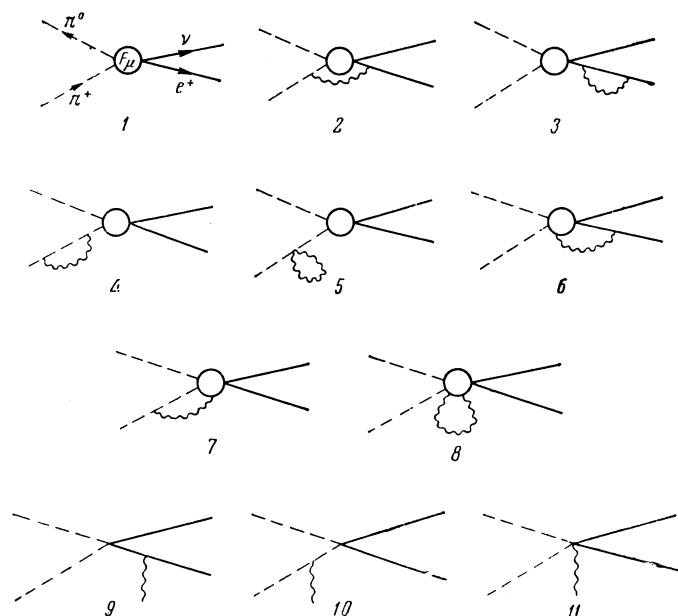
arguments following Eq. (13), we find again that in the case under consideration $a(0) = 1$.

The absence in the β decay of effects linear in mass differences was earlier established by Behrends and Sirlin^[3], but their proof is less general and is not applicable to the $\pi^+ \rightarrow e^+ + \nu + \pi^0$ decay and to hyperon decays.

2. THE $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ DECAY

1. The $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay has recently been found experimentally (cf. [4-6]). The decay was first predicted by Zel'dovich^[7] in 1954. The study of this process enables us in principle to test the current vector conservation hypothesis in weak interactions,^[1,2] since it follows from this hypothesis that the interaction constant of the process is equal to the constant of the vector part of the β decay interaction. It is interesting to find out how accurate is the equality of the constants, taking into consideration the approximate character of the vector current conservation.

2. Let us estimate the radiative corrections to the $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay. In addition to the main process (see diagram 1 in the figure) we have a set of processes involving the emission of a virtual photon (diagram 2-8) and also the decay with the emission of a bremsstrahlung photon (diagram 9-11). In diagrams 9-11 we neglect the π -meson form factor. Let us first consider the case where the weak interaction form factor is independent of q^2 (the square of the difference of the meson four-momenta) and of the difference in meson mass. We are then left with diagrams 1 to 7.



²⁾We use the well-known expression for the matrix elements of rotation generators $t^{(i)}$ in canonical basis:

$$\langle m \pm 1 | t^{(1)} \pm it^{(2)} | m \rangle = \sqrt{t(t+1) - m(m \pm 1)}.$$

The resulting set of diagrams is generated in the lowest order by the following interaction Lagrangian

$$L_{int} = -e\bar{\Psi}_e \hat{A} \Psi_e + ie \left(\frac{\partial \varphi^*}{\partial x_\mu} \varphi - \varphi^* \frac{\partial \varphi}{\partial x_\mu} \right) A_\mu + e^2 A_\mu A^\mu \varphi^* \varphi + \left[iG \left(\varphi^* \frac{\partial \varphi_0}{\partial x_\mu} - \frac{\partial \varphi^*}{\partial x_\mu} \varphi_0 \right) l_\mu - eG\varphi^* \varphi_0 l_\mu A^\mu + \text{comp. conj.} \right] \quad (17)$$

where $l_\mu = \bar{\Psi}_\nu \gamma_\mu (1 + \gamma_5) \Psi_e$, $G = (1.01 \pm 0.01) 10^{-5}/M^2$, and M is the nucleon mass. The contribution of such diagrams was calculated in [8], in which, however, the authors committed a number of errors. The diagrams were used with incorrect relative signs, and were therefore on the whole not invariant with respect to gauge transformation. An error was also made in calculating the contribution of diagram 4 and of the bremsstrahlung photon (diagrams 9–11).

We shall calculate the positron spectrum in the case where the decay with the bremsstrahlung photon emission is not singled out experimentally:

$$W(\theta) d\theta = G^2 m_e^5 \pi^{-3} (1 + 2m_{\mu^+}^{-1} \text{ch } \theta) (1 + \Phi_1) \times (\text{ch } \theta_0 - \text{ch } \theta)^2 \text{sh}^2 \theta \text{ch } \theta d\theta; \\ \Phi_1 = -\alpha [\pi \text{sh } \theta \text{ch } \theta]^{-1} [\theta + \theta \text{ch}^2 \theta + 2\theta^2 \text{ch}^2 \theta - 3 \text{sh } \theta \text{ch } \theta - \frac{3}{2} \text{sh } \theta \text{ch } \theta \ln (\Lambda/m_e) + 2 \text{ch } \theta (\text{sh } \theta - \theta \text{ch } \theta) \times \ln [2 (\text{ch } \theta_0 - \text{ch } \theta)] 2 \text{ch}^2 \theta / (1 - e^{-2\theta}) + \frac{1}{3} (\text{ch } \theta_0 - \text{ch } \theta) \times (2 \text{sh } \theta - \frac{7}{4} \theta \text{ch } \theta - \frac{1}{4} \theta \text{ch } \theta_0)]. \quad (18)^*$$

In this expression m_e and μ_+ are the positron and π^+ meson mass, respectively, and $\cosh \theta = \epsilon/m_e$, where ϵ is the positron energy. θ varies within the limits $0 \leq \theta \leq \theta_0$, and θ_0 is given by the relation

$$\text{ch } \theta_0 = \epsilon_{max}/m_e = \Delta/m_e (1 - \Delta/2\mu_+)$$

where Δ is the mass difference of the π^+ and π^0 mesons, Λ is the ultraviolet cutoff, and $f(x) = -\int_0^x [\ln(1-t) dt]/t$ is the Spence function. The function Φ_1 is due to the radiative corrections and is very similar to the expressions for radiative corrections to the electron spectrum in β decay [Kinoshita and Sirlin, [9] Eq. (4.2) ³⁾].

³⁾For a comparison, we can first simplify Eq. (4.2) in [9] taking into consideration that the complete set of Spence functions in Eq. (4.2) can be in fact reduced to one function:

$$f(\beta) - f(-\beta) + f\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2} f\left(\frac{1-\beta}{2}\right) - \frac{1}{2} f\left(\frac{1+\beta}{2}\right) = 2f\left(\frac{2\beta}{1+\beta}\right) - \ln \frac{2}{1+\beta} \text{arctanh } \beta = 2f(1 - e^{-2\theta}) - \theta \ln(2e^{-\theta} \text{sh } \theta)$$

and the term with $\ln(2e^{-\theta} \sinh \theta)$ cancels the last term in Eq. 4.2 in [9].

*ch = cosh; sh = sinh.

In the case where the experimental resolution of the photon energy ω_0 is small ($\omega_0 \ll m_e$), the function Φ_1 in Eq. (18) should be exchanged for Φ_2 :

$$\Phi_2 = -\alpha [\pi \text{sh } \theta \text{ch } \theta]^{-1} \times [\theta - 2\theta \text{ch}^2 \theta + 2\theta^2 \text{ch}^2 \theta + 2 \text{ch } \theta (\text{sh } \theta - \theta \text{ch } \theta) \times \ln(2\omega_0/m_e) 2 \text{ch}^2 \theta / (1 - e^{-2\theta}) - \frac{3}{2} \text{sh } \theta \text{ch } \theta \ln \left(\frac{\Lambda}{m_e} \right)]. \quad (19)$$

The integration in Eq. (18) leads to the following expression for the decay probability:

$$w = w_0 (1 - \frac{3}{2} \Delta/\mu_+ - 5m_e^2/\Delta^2 + \delta_1), \quad (20)$$

where $w_0 = G^2 \Delta^5 / 30\pi^3$ and $\delta_1 = (3\alpha/2\pi) [\ln(\Lambda/2\Delta) - 4\pi^2/9 + 59/20]$ arise because of the function Φ_1 . When exchanging Φ_1 for Φ_2 in Eq. (20), it is necessary to exchange δ_1 for δ_2 , where

$$\delta_2 = \frac{\alpha}{\pi} \left(\frac{3}{2} \ln \frac{\Lambda}{m_e} + \frac{47}{60} \ln \frac{2\Delta}{m_e} + 2 \left(1 - \ln \frac{2\Delta}{m_e} + \frac{47}{60} \right) \ln \frac{\Lambda}{\omega_0} - \frac{\pi^2}{3} - \frac{2899}{900} \right).$$

δ_1 and δ_2 are of the same order of magnitude (for $\omega_0 \sim 1/10 m_e$).

In the following we consider only δ_1 . Numerically $\delta_1 = 0.012$ for $\Lambda = M$. Kinematical corrections [second and third terms in Eq. (20)] amount to -0.113 . As the result, the decay probability is found to be

$$w = 3.95 \cdot 10^{-1} \text{ sec}^{-1}. \quad (21)$$

The error of the known value of the mass difference of the π^+ and π^0 mesons [$\Delta = (4.59 \pm 0.01) \text{ MeV}$] causes an error of 1.5% in the decay probability. The variation of Λ by a factor of 10 leads to 0.8% change in w . An attempt to cut off the interaction with the photon in the positron-pion vertices at different Λ (similar to what was done for β decay in [10]) assuming that the electron is cut off in weak interactions at $\Lambda_e \sim 300 \text{ BeV}$, and the meson in strong interactions at $\Lambda_\pi = M \sim 1 \text{ BeV}$, leads to the exchange in Eq. (20) of Λ for $\Lambda_{eff} \approx 2.5 \Lambda = 2.5 M$, ⁴⁾ and we obtain practically the same results as earlier for $\Lambda = \Lambda_e = \Lambda_\pi = M$.

Thus, the radiative corrections and the errors in Eq. (20) are very small. There is, however, a source of an additional error, which we have not considered so far.

3. If we take the weak interaction form factor

⁴⁾Such a result is obtained if we choose the cut-off factor in the positron vertex in the form $\Lambda_e/[\Lambda_e^2 + k^2]^{1/2}$ and in the π meson vertex in the form $\Lambda_\pi/[\Lambda_\pi^2 + k^2]^{1/2}$. It is then necessary to make use of the fact that $\Lambda_e \gg \Lambda_\pi$.

into account, then the matrix element corresponding to diagram 1 in the figure is proportional to the expression [compare Eqs. (4) and (5)]

$$GF_{\mu} \bar{u} \gamma^{\mu} (1 + \gamma_5) v_e, \quad (22)$$

$$F_{\mu} = f(q^2) (p^+ + p^0)_{\mu} + h(q^2) q_{\mu}. \quad (23)$$

We can neglect the second term in Eq. (23) (see footnote ¹⁾). In the theory with conserved vector current the quantity $f(q^2)$ coincides, with an accuracy to the meson mass difference, with the π -meson electromagnetic form factor. We can assume that $f(q^2) \equiv f(q^2/\mu^2) \approx f(0)$ since $q^2/\mu^2 \sim \Delta^2/\mu^2 \sim 10^{-3}$. The theorem about the absence of effects linear in the mass difference (see section 3.1) enables us to put $f(0) = 1 + O(\Delta^2/\mu^2)$. Thus, taking the form factor in diagram 1 into account does not lead to a considerable uncertainty in the decay probability.

4. The source of an additional error lies in an effect analogous to that discussed in ^[10] for the radiative corrections to the β decay. In the presence of form factors, the diagrams 2—8 are gauge-invariant and processes 6—8 occur therefore in addition to those discussed by us in Sec. 2.2. (It should be noted that the cases of a local interaction of five fields, contained in diagrams 6 and 7, were also taken into account in Sec. 2.) Let us denote by $F_{\mu}^{\lambda}(p^+, p^0, q, k)$ and $F_{\mu}^{\lambda\lambda'}(p^+, p^0, q, k, k')$ the form factors involving the emission, directly from F_{μ} in diagram 1, of one and two photons respectively, with momenta k and k' and polarizations λ and λ' . We have also the following identities of the Ward type for F_{μ}^{λ} and $F_{\mu}^{\lambda\lambda'}$:

$$k_{\lambda} F_{\mu}^{\lambda}(p^+, p^0, q, k) = e [F_{\mu}(p^+ - k, p^0, q - k) - F_{\mu}(p^+, p^0, q)], \quad (24)$$

$$k_{\lambda} k_{\lambda'} F_{\mu}^{\lambda\lambda'}(p^+, p^0, q, k, -k) = e^2 [F_{\mu}(p^+ + k, p^0, q + k) - F_{\mu}(p^+ - k, p^0, q - k) - 2F_{\mu}(p^+, p^0, q)]. \quad (25)$$

The proof of these relations obtained by summing the perturbation theory diagrams, was given in ^[10]. ⁵⁾ The contribution of diagrams 6—8 can be estimated as follows: Eqs. (24) and (25) enable us to express the quantities F_{μ}^{λ} and $F_{\mu}^{\lambda\lambda'}$, for k

and k' equal to zero, through the derivatives of the form factor F_{μ} . Thus, e.g.,

$$F_{\mu}^{\lambda}|_{k=0} = -e g_{\lambda\mu}, \quad (26)$$

$$F_{\mu}^{\lambda\lambda'}|_{k=k'=0} = -2e^2 (p^+ + p^0)_{\mu} g_{\lambda\lambda'}/\mu^2. \quad (27)$$

In calculating the contribution from diagrams 6—8 we used Eqs. (26) and (27) and cut off the integration over the photon momentum at the nucleon mass. The resulting correction to the decay probability equals $\sim 0.3\%$. Strictly speaking this quantity is actually not a correction but characterizes only the magnitude of additional errors, since the above discussion does not pretend to great accuracy.

5. Summarizing the results of the preceding sections we come to the conclusion that the probability of the $\pi^+ \rightarrow \pi^0 + e^+ + \nu$ decay is given by Eq. (21) with an error of 2.5—3%. Moreover, a greater part of this error (1.5%) is due to the errors in the experimental value of the π^+ and π^0 meson mass difference [$\Delta = (4.59 \pm 0.01)$ MeV].

The probability considered usually is that of the $\pi^+ \rightarrow \mu^+ + \nu$ decay, which essentially determines the lifetime of the π^+ meson. The ratio R is found to be

$$R = (1.01 \pm 0.03) \cdot 10^{-8}. \quad (28)$$

For comparison let us quote the values of R obtained recently by three different experimental groups:

R	Experimental group
$(1.1 \pm 1.0) \cdot 10^{-8}$	Dubna ^[4]
$(1.7 \pm 0.5) \cdot 10^{-8}$	Geneva ^[5]
$(2.0 \pm 0.6) \cdot 10^{-8}$	Berkeley ^[6]

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EFFECT OF JOULE HEAT ON THE QUENCHING OF SUPERCONDUCTIVITY BY A CURRENT

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The alteration in the form of the temperature and current dependences of the resistance of superconducting cylindrical samples resulting from internal Joule heating is considered. From the results thus obtained it is possible to explain the departure of the experimental data from the well-known London formula (1) [1].

THE current and temperature dependence of the resistance R of a cylindrical superconducting specimen was treated theoretically by London as long ago as 1937 [1]. The relation

$$R = 0 \quad \text{for } I < I_c, \\ R = \frac{1}{2} R_n [1 + \sqrt{1 - (I/I_c)^2}] \quad \text{for } I > I_c, \quad (1)$$

was derived, where I is the current in the sample, I_c is the critical current at the given temperature, and R is the resistance of the sample in the normal state.

The derivation of this relation is based upon the assumption that for $I > I_c$ an interior portion of the sample, whose radius tends to zero as the current increases, is in the intermediate state. The structure of the intermediate state must consist of alternating normal and superconducting layers perpendicular to the axis of the cylinder; the thickness of the normal layers increases linearly with distance from the axis, but they do not completely fill the length of the cylinder at the boundary of the intermediate region. Experimental investigations [2,3] have not contradicted this model of the intermediate state. As regards the relations (1), measurements (cf. [3-8]), despite qualitative agreement, have revealed systematic departures from the curve (1): the jump in resistance at $I = I_c$ is found to exceed $0.5 R_n$; R reaches R_n at finite values of the current, and the dependence of R upon I shows hysteresis.

Various hypotheses have been advanced in order to explain these discrepancies: distortion of the intermediate state structure due to surface tension at the interfaces between the normal and superconducting layers [6], or a change in the resistivity of the sample due to scattering of electrons at these boundaries [9]; a dynamical model of the intermediate state differing from that just described has also been suggested [10]. The majority

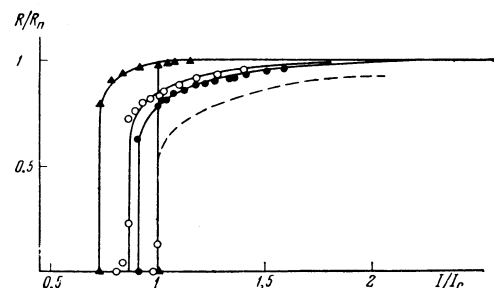
of these authors have failed to associate these discrepancies with the effects of Joule heating; that this is, however, their source has recently been demonstrated by Trořnar [3]. A systematic accounting for this effect makes it possible to explain to a considerable degree the divergence of theory and experiment.

If the resistance per unit length of the sample is known as a function of temperature and current, $R = F(T, I)$, then for a given current and helium bath temperature T_b the measured value of the resistance R will be determined by the equation

$$R = F(T_b + I^2 R / 2\pi a h, I), \quad (2)$$

where h is the coefficient for heat transfer through the sample-liquid helium boundary, and a is the cross-sectional radius of the sample. We consider the temperature within the specimen to be constant, since the thermal resistance of a sample of the usual dimensions is much less than $(1/2)\pi a h$; we neglect the heat transfer through the ends of the specimen.

Solving Eq. (2), we transform the functional dependence of R upon (T_b, I) into the experimentally observable dependence of R upon (T_b, I) . The



Dependence of resistance upon current for various specimens of tin: Δ —No. 12 [3], $T_b = 2.87^\circ \text{K}$; $\circ$ —No. 12 [3], $T_b = 1.85^\circ \text{K}$; $\bullet$ —No. 6 [7], $T_b = 3.27^\circ \text{K}$. The solid curves are constructed according to Eq. (3); the dashed curves represent London's Eq. (1). [1]

solution to Eq. (2) is determined from the aggregate of points of intersection between the surface $R = F(T, I)$ and the family of surfaces $R = 2\pi ahI^{-2}(T - T_b)$; choosing either I or T_b , we obtain the observable transition curves with respect to either current or temperature. By differentiating (2), we find that if at any given intersection point $\partial R/\partial T > 2\pi ah/I^2$, then $dR/dI < 0$ and $dR/dT_b < 0$ and, in consequence, hysteresis will be observed along the corresponding transition curves.

In the present case, proceeding from (1), we arrive at the relations

$$R = 0 \quad \text{for} \quad 0 < I < I_c,$$

$$R = \frac{R_n}{2 + (I/I_c)^2 \delta^2/2} \times \left[1 + \frac{\delta}{2} + \sqrt{1 + \delta - (I_c/I)^2} \right] \quad \text{for} \quad \frac{I_c}{\sqrt{1 + \delta}} < I < \frac{I}{\sqrt{\delta}},$$

$$R = R_n \quad \text{for} \quad I > I_c/\sqrt{\delta}. \quad (3)$$

Here

$$\delta = H_c |dH_c/dT| \rho / 8\pi^2 ha,$$

ρ is the resistivity of the normal phase, and H_c is the critical field ($H_c = 2I_c/a$).

In the derivation of Eqs. (3) it was assumed that the change in the magnitude of the critical field ΔH_c and the temperature rise of the specimen relative to the helium bath ΔT are linearly related: $\Delta H_c = (dH_c/dT)\Delta T$, and that h is independent of ΔT (the latter condition will prevail for sufficiently large thermal loadings both in He II^[11] and in He I^[8]). The quantity δ has a simple physical significance: $\delta \sim \Delta T/(T_c - T_b)$ (T_c is the critical temperature of the specimen).

We see that in the region $[I_c/\sqrt{1 + \delta}, I_c]$ the dependence is not single-valued; i.e., hysteresis arises. The upward jump in the resistance takes place at $I = I_c$, the return at $I = I_c/\sqrt{1 + \delta}$. The full resistance is present for a current $I_n = I_c/\sqrt{\delta}$. If, for example, the initial jump in resistance is taken to be $0.8 R_n$, then $I_n = 2I_c$, in full agreement with the facts as established long since by Shubnikov and Alekseevskii^[4].

The figure shows $R(I)$ curves plotted for various samples according to Eq. (3). The parameter h as well as δ , which depends upon the experimental conditions and the surface state of the sample, has been selected appropriately for each case. For tin with a relatively high residual resistance, greater than $\rho \sim 10^{-8} \Omega \text{cm}$, the values of h obtained in this way agree for various samples, and amount, in He I, to $0.07 - 0.13 \text{ W cm}^{-2} \text{ deg}^{-1}$, and in He II, to $0.6 - 0.8 \text{ W cm}^{-2} \text{ deg}^{-1}$ (cf. the table).

For He II ($T_b \ll 2.19^\circ \text{K}$) these results agree

Sample	$T_b, ^\circ\text{K}$	δ	$h, \text{ W cm}^{-2} \text{ deg}^{-1}$
№ 12 [3]	3.47	0.6	0.09
	2.87	0.9	0.13
	1.85	0.32	0.67
№ 5 [7]	3.48	0.14	0.09
№ 6 [7]	3.27	0.20	0.07
	1.95	0.25	~ 0.8

well with the data from direct measurements^[11]. In the case of He I, where the heat evolved depends strongly upon the experimental circumstances, the data obtained lie within the range of established values for h ^[8,12]. It may therefore be suggested that for such samples, the departures from curve (1) are almost entirely due to the effects of Joule heating.

For tin samples of higher purity ($\rho \sim 10^{-9} \Omega \text{cm}$; for example, Nos. 1–5^[3] and 7^[7]), however, the resistance values calculated from Eq. (3) do not agree as closely with the experimental data as for specimens with larger residual resistances. This means that the isothermal transition curve for “pure” specimens departs somewhat from the transition curve defined by London’s equation (1). This departure evidently occurs when the mean free path of the electrons in the normal state becomes greater than the periodicity of the intermediate state structure, which can lead (see^[9]) to an additional increase in the resistance due to the scattering of electrons at the domain boundaries.

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EFFECTS OF INELASTIC PROCESSES ON ELECTRON-ELECTRON AND ELECTRON-POSITRON SCATTERING

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By means of dispersion relations it is shown that the effect of inelastic processes, including the production of strongly interacting particles, is small and cannot significantly affect the analysis of experiments made to test the validity of electrodynamics by the use of colliding beams.

1. Experiments with colliding beams of electrons and positrons at high energies, which are opening up new possibilities in the physics of unstable particles, are often regarded as a possible way of testing experimentally the validity of the laws of quantum electrodynamics. One of the processes most susceptible to analysis is the elastic scattering of electrons by positrons and of electrons by electrons (e^+e^+ , e^-e^-), or, in general, ee scattering. The analysis of future experimental data on elastic ee scattering from the point of view of the validity of electrodynamics must, however, be preceded by a study of the inelastic processes which occur simultaneously with the elastic ones, and an investigation of the effect of the inelastic processes on the elastic processes.

Here we would like to examine what effects on the real part of the amplitude for elastic ee scattering are produced by certain processes which are not usually considered in the framework of the electrodynamics of electrons and photons. Because there are no experimental data, in estimating the effects of the inelastic processes we shall take for the cross sections of these processes the expressions found for them in the one-photon approximation.

The calculation of the actual effects of the inelastic processes is made below by means of the method of dispersion relations (d. r.). The complete d. r. for ee scattering are not considered. As a first step we include the contribution to the real part of the scattering amplitude from dispersion terms of the form

$$\delta D(\omega_0) = \frac{k_0^2}{4\pi^2} \int_{\omega_t}^{\infty} \frac{d\omega}{k} \left[\frac{\sigma_+(\omega)}{\omega - \omega_0} + \frac{\sigma_-(\omega)}{\omega + \omega_0} \right], \quad (1)$$

where, for example, for e^+e^- (e^-e^-) scattering σ_+

is the total cross section for the e^+e^- (e^-e^-) interaction, and σ_- is that for the e^-e^- (e^+e^-) interaction. The nondispersion part of the amplitude is to be added to the expression (1), which is the d. r. for forward scattering in a system in which one of the particles is at rest, and which near threshold gives the effect of inelastic interactions for arbitrary angles.

2. We consider here effects near threshold which are due to the production of muon pairs in positron-electron collisions:

$$e^+ + e^- \rightarrow \mu^+ + \mu^-. \quad (2)$$

Near its threshold the existence of process (2) leads to an effect^[1,2] in e^+e^- scattering, and we shall estimate the magnitude and energy width of this effect. In electron-electron scattering in the range of energies close to the threshold of the reaction (2) the derivative of the cross section with respect to the energy of course does not become infinite at the threshold. In the framework of the d. r. method, however, the process (2) leads through its contribution to the cross-symmetric process to some effects in e^-e^- scattering, and we shall also obtain estimates of these effects.

An expression for the cross section of process (2) has been obtained by Berestetskii (cf. e. g., discussion in^[3]):

$$\sigma = \frac{2\pi}{3} r_0^2 \frac{m}{\omega} \left(1 + \frac{\omega_t}{2\omega} \right) \left(1 - \frac{\omega_t}{\omega} \right)^{1/2}, \quad (3)$$

where $\omega_t \approx 2m_\mu^2/m = 4 \times 10^{10}$ eV is the threshold of reaction (2) in the system where one of the electrons was at rest before the collision; m_μ and m are the masses of the muon and electron; and r_0 is the classical electron radius.

Substitution of Eq. (3) in Eq. (1) gives as the expression for the dispersion part of the amplitude

of e^+e^- scattering caused by process (2)

$$\delta D_+(\omega_0) = \frac{1}{6\pi} \left(\frac{e^2}{\hbar c} \right)^2 \frac{k_0 c}{\omega_0} \frac{k_0 c}{\omega_t} \frac{\hbar c}{mc^2} \frac{\Lambda_+(y_0)}{y_0}, \quad (4)$$

where $y_0 = \omega_t/\omega_0$, and

$$\begin{aligned} \Lambda_+ &= y_0 \left\{ \frac{28}{15} + (y_0 - 1) \left(\frac{8}{3} + y_0 \right) \right. \\ &\quad \left. - y_0 (2 + y_0) (y_0 - 1)^{1/2} \arctg (y_0 - 1)^{1/2} \right\} \text{ for } y_0 \geq 1, \\ \Lambda_+ &= y_0 \left\{ \frac{28}{15} - (1 - y_0) \left(\frac{8}{3} + y_0 \right) \right. \\ &\quad \left. + y_0 (2 + y_0) (1 - y_0)^{1/2} \ln \left| \frac{1 + \sqrt{1 - y_0}}{1 - \sqrt{1 - y_0}} \right| \right\} \text{ for } y_0 \leq 1. \end{aligned} \quad (4')^*$$

The value reached by the threshold "perturbation" of the real part of the amplitude is

$$\delta D_+(\omega_t) = \frac{1}{3\pi} \frac{14}{15} \frac{e^2}{\hbar c} r_0. \quad (5)$$

Since the magnitude of the cross section for e^+e^- scattering in the one-photon approximation falls off sharply with increasing energy, and its value when the energy of each of the colliding particles is $E_b = m_\mu$ (and for $\theta = 90^\circ$) is about

$$\sigma_s(90^\circ, m_\mu) = \frac{9}{16} r_0^2 (m/m_\mu)^2, \quad (6)$$

one might think that the additional factor $e^2/\hbar c$ in Eq. (5) can be compensated by the ratio of the muon and electron masses. It is not the value (5), however, that is to be compared with the value of the cross section (6) in the c. m. s. (for colliding beams this is the laboratory system), but the result of transforming the quantities (4) and (5) to the c. m. s.

The values of the forward scattering amplitude A_L in the l. s. (system in which one particle was at rest before the collision) and that in the c. m. s. (A_C) are connected by the relation

$$A_L/A_C = k_L/k_C \approx 2E_b/m, \quad (7)$$

where the last equality holds for high energies. We finally have to compare the threshold perturbation in the system of the colliding beams

$$\delta D_+^C(\omega_t) = \frac{m}{m_\mu} \frac{1}{6\pi} r_0 \frac{e^2}{\hbar c} \quad (8)$$

with the approximate value

$$\sqrt{\sigma_s(90^\circ, m_\mu)} = \frac{3}{4} r_0 m/m_\mu.$$

Thus the fractional effect amounts to

$$\delta D_+^C(\omega_t)/\sqrt{\sigma_s(90^\circ, m_\mu)} = \frac{2}{9} \frac{e^2}{\pi \hbar c}, \quad (9)$$

and the mass ratio has dropped out, so that the maximum effect on the cross section at $\theta = 90^\circ$

does not exceed 0.1 percent. We note that the energy width below threshold of the effect near the threshold is given by Eq. (4) as

$$\Delta E \equiv \sqrt{\Delta(E_b^2)} = 22.4 \text{ MeV}.$$

The contribution (3) to the real part of the amplitude for e^-e^- scattering is due to the second term in Eq. (1). For this case use of Eq. (3) together with Eqs. (4) and (5) leads to expressions in which $\Lambda_+(y_0)$ is replaced by the quantity

$$\frac{\Lambda_-}{y_0} = \frac{1}{3} - 2 \left(1 - \frac{y_0}{2} \right) \left[1 - \frac{\sqrt{1+y_0}}{2} \ln \frac{(1 + \sqrt{1+y_0})^2}{y_0} \right], \quad (10)$$

which describes a contribution positive in sign and monotonically increasing with the energy. The ratio of the threshold perturbations is an appreciable quantity:

$$\Lambda_-(\omega_t) = 0.29 \Lambda_+(\omega_t). \quad (11)$$

The effect of process (2) on e^-e^- scattering increases with increasing energy.

Near thresholds for production of other fermion pairs Eq. (8) can be altered by the strong interaction in the final state of the inelastic reaction. For $\bar{\Lambda}\Sigma^0$ pairs this remains valid for the relative parity $P(\Lambda\Sigma) = +1$.

3. On this dispersion effect associated with the process (2) there are superimposed effects associated with other processes. The inelastic process with lowest threshold which leads in the one-photon approximation to the production of neutral pions is

$$e^- + e^+ \rightarrow \pi^0 + \gamma \quad (12)$$

and its total cross section (in cm^2) is [4]

$$\sigma = 2.75 \cdot 10^{-32} |G(-K^2)/G(0)|^2 (1 - x^2)^{3/2} \quad (x = m_\pi/2E_b), \quad (13)$$

where according to an estimate by Cabibbo and Gatto the ratio of the form-factors can reach the value

$$|G(-K^2)/G(0)|^2 \approx 250.$$

Substitution of Eq. (13) in Eq. (1) gives as the formulas relating to e^+e^- scattering (δD_+) and e^-e^- scattering (δD_-) (with $y_0 = \omega_t/\omega_0$)

$$\delta D_\pm(\omega_0) = \frac{k_0^2}{4\pi^2} \left| \frac{G(-K^2)}{G(0)} \right|^2 \frac{2.75 \cdot 10^{-32}}{\omega_t} y_0 \cdot 2\Lambda_\pm(\omega_0), \quad (14)$$

where for e^+e^- scattering

$$\begin{aligned} \Lambda_+ &= \frac{1}{3} - (y_0 - 1) + (y_0 - 1)^{1/2} \arctg (y_0 - 1)^{1/2}, \text{ for } y_0 \geq 1, \\ \Lambda_+ &= \frac{1}{3} + (-y_0 + 1) \\ &\quad - \frac{(1 - y_0)^{3/2}}{2} \ln \left| \frac{1 + \sqrt{1 - y_0}}{1 - \sqrt{1 - y_0}} \right|, \text{ for } y_0 \leq 1 \end{aligned} \quad (14')$$

and for e^-e^- scattering

* $\arctg = \tan^{-1}$.

$$\Lambda_- = \frac{(1+y_0)^{3/2}}{2} \ln \left| \frac{1 + \sqrt{1+y_0}}{\sqrt{1+y_0}-1} \right| - \frac{1}{3} - (y_0 + 1). \quad (15)$$

From Eqs. (14), (15), (7), (8), and (9) one finds that the contributions of process (12) for the two elastic scattering processes are comparable and very small. The conclusion that the effect of process (12) is small (< 0.1 per cent) remains true when the influence of the form-factor is taken into account.

Even if we suppose that in the production of charged-pion pairs

$$e^- + e^+ \rightarrow \pi^- + \pi^+ \quad (16)$$

the presence of the form-factor in the cross section (cm²)

$$\sigma = 0.53 \cdot 10^{-32} m_\pi^2 z^{-2} (1 - z^{-2})^{3/2} |F(-4E_b^2)|^2 \quad (17)$$

(m_π in BeV, $z = E_b/m_\pi$) leads to a further factor of the order of 20,^[4] the magnitude of the perturbation is

$$\delta D_\pm(\omega_0) = \frac{k_0^2}{4\pi^2} \frac{5 \cdot 10^{-30}}{\omega_t} \Lambda_\pm(y_0), \quad (18)$$

where for e^+e^- scattering

$$\begin{aligned} \Lambda_+ = 2y_0 \left\{ y_0 \left[\frac{1}{3} - (y_0 - 1) + (y_0 - 1)^{3/2} \arctg(y_0 - 1)^{1/2} \right] - \frac{1}{5} \right\} & \text{for } y_0 \geq 1, \\ \Lambda_+ = 2y_0 \left\{ y_0 \left[\frac{1}{3} + (1 - y_0) - \frac{(1 - y_0)^{3/2}}{2} \ln \left| \frac{1 + (1 - y_0)^{1/2}}{1 - (1 - y_0)^{1/2}} \right| \right] - \frac{1}{5} \right\} & \text{for } y_0 \leq 1, \end{aligned} \quad (19)$$

and for e^-e^- scattering

$$\begin{aligned} \Lambda_- = 2y_0 \left\{ y_0 \left[\frac{1}{3} + (1 + y_0) - \frac{(1 + y_0)^{3/2}}{2} \ln \left| \frac{1 + (1 + y_0)^{1/2}}{(1 + y_0)^{1/2} - 1} \right| \right] - \frac{1}{5} \right\} \end{aligned} \quad (20)$$

and remains a small quantity. At threshold $\Lambda_-(\omega_t) = 0.21\Lambda_+(\omega_t)$, and this increases to $\Lambda_- = 6\Lambda_+$ for $\omega_0 = 2\omega_t$.

Thus we can conclude that inelastic processes which include strong interactions cannot have much effect on the results of experiments to test the validity of quantum electrodynamics by the use of colliding beams. The total effect does not exceed 1 per cent.

The d. r. method used here can also be useful for more detailed calculations when experimental data on the inelastic processes have been obtained, and also for the analysis of ee collisions at smaller energies.

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DIRECTED COHERENT RADIATION DUE TO BREAKDOWN NEAR THE TRACK OF AN IONIZING PARTICLE

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Directed emission of radio waves produced as a result of breakdown near the track of an ionizing particle in a medium located in an electric field is considered. It is shown that a strong asymmetry of the radiation arises if the particle velocity is comparable to or exceeds the propagation velocity of the radio waves in the medium. It is pointed out that the effect can be used to obtain sharply directed radiation and also to determine the velocity of the particle initiating the breakdown by measuring the angle of maximum emission.

INTRODUCTION

THE well-known forms of radiations which arise as a result of the motion of a particle in a medium (for example, Cerenkov radiation, radiation by heterogeneities of a medium etc.) draw energy from a moving particle and have comparatively low intensity. It seemed interesting to us to examine radiation effects which, although still produced by charged particles, draw their energy from external fields.

One of the possible versions of this effect is the radiation which arises as a result of the breakdown near the track of an ionizing particle in a medium with an electric field the intensity of which is close to breakdown. Easier conditions for the breakdown and its localization near the track can be ensured, for example, by the use of two-phase media (porous media, media with voids and bubbles filled with a special gas, blocks of corrugated plates which form closed spaces between the plates, etc.) and also by the use of media with inclusions that have a smaller breakdown strength (semiconductor inclusions, semiconductors with impurities, etc.). Radiation from pre-breakdown currents or from currents at the start of breakdown, initiated by the particle, can be observed also in a one-phase medium (gas, liquid, semiconductor, etc.).

We consider the properties of coherent radiation due to breakdown near the track.

COHERENT RADIO EMISSION DUE TO BREAKDOWN NEAR THE TRACK OF A PARTICLE:

Let us assume that as a particle moves through a medium localized breakdowns are formed near

the particle's track under the action of a strong external electric field. It is obvious that one can speak of coherence of the radiation in this case if the statistical delay of the breakdowns following the flight of the particle through a given point of a medium is small in comparison with the period of the propagated waves. Experiments with parallel-plate spark counters (see, for example, [1,2]) have demonstrated that the spread of the delay and the time of buildup to half intensity of the spark is of the order of a fraction of a nanosecond for the size of sparks of interest to us. This shows that coherent effects can be expected in the decimeter band. It is precisely for these waves that a medium with small heterogeneities will not act like a homogeneous medium. We assume henceforth that the requirements for coherence can be ensured and that the medium is homogeneous for radio waves. Coherent generation of micro-wave and optical radiation as a result of breakdown near the track is possible in semiconductors.

If the distance from the point of reception exceeds the length of the tract segment along which the conditions for breakdown are realized, then one can use the expression for the wave field of a system of dipoles:*

$$E(t) = \frac{1}{c^2 R_0} \left[\int d\ddot{\mathbf{P}} \left(t - \frac{R}{c} \right) \mathbf{n} \right] \mathbf{n},$$

$$\mathbf{H}(t) = \frac{V_{\varepsilon}}{c^2 R_0} \left[\int d\ddot{\mathbf{P}} \left(t - \frac{R}{c} \right) \mathbf{n} \right],$$

where $R = R_0 - \mathbf{r} \cdot \mathbf{n}$, $d\ddot{\mathbf{P}} = \ddot{\mathbf{P}}_1 dz$, and $\mathbf{P}_1(t)$ is the resulting dipole moment per unit breakdown-current track length.

* $[\mathbf{Pn}] = \mathbf{P} \times \mathbf{n}$.

Transforming to Fourier components and integrating along the track of the particle we obtain the spectral component of the wave field

$$E_{\omega} = \frac{1}{c^2 R_0} e^{ik'R_0} L \frac{2 \sin \{k' (c'/v - \cos \theta) L\}}{k' (c'/v - \cos \theta) L} [(\ddot{\mathbf{P}}_1)_{\omega} \mathbf{n}],$$

where $k' = \omega/c'$, v is the speed of the particle, c' is the speed of propagation of the radio waves, L is the length of the track segment, and θ is the angle between the track and the direction of observation. The appearance of an abrupt asymmetry follows already from the above formula with $v \gtrsim c'$ for $\cos \theta = c'/v$ (the sharpness of the directivity is $\Delta \theta \sim \pi/k' L \sin \theta$).

In the case of a localized breakdown as the result of the abrupt occurrence of dipole moments we have

$$(\ddot{\mathbf{P}})_{\omega} = i\omega (\dot{\mathbf{P}})_{\omega} \approx i\omega \mathbf{P}_{max}/2\pi \text{ at } \omega\tau \ll 1,$$

where τ is the breakdown time.

In the case of an exponential growth of the breakdown current we have $J_1 = \dot{\mathbf{P}}_1 = J_{10} e^{\alpha t}$ for t in the interval from 0 to τ and $J_1 = 0$ for all remaining t . Then we obtain

$$(\ddot{\mathbf{P}})_{\omega} = \frac{i\omega}{2\pi} \frac{J_{10}}{\alpha + i\omega} \{e^{(\alpha + i\omega)\tau} - 1\}.$$

The spectral intensity of the radiation is

$$\mathcal{E}_{\omega} = \frac{\sqrt{\varepsilon} \omega^2 J_{10}}{2\pi^2 c^3 R_0^2} \frac{L^2}{\alpha^2 + \omega^2} \frac{\sin^2 \kappa L}{(\kappa L)^2} \left\{ (e^{\alpha\tau} - 1)^2 + 4e^{\alpha\tau} \sin^2 \frac{\omega\tau}{2} \right\} \sin^2 \varphi,$$

where $\kappa = k' (c'/v - \cos \theta)$ and φ is the angle between the direction of the breakdown field and the direction of the reception. At $\alpha\tau \gg 1$ and $\omega\tau \ll 1$ we obtain the spectral intensity of radiation

$$\mathcal{E}_{\omega} \approx \frac{\sqrt{\varepsilon}}{2\pi^2 c^3} \omega^2 \frac{L^2}{R_0^2} \frac{\sin^2 \kappa L}{(\kappa L)^2} P_{10}^2 \sin^2 \varphi,$$

where P_{10} is the maximum value of the linear dipole moment. For example, if the breakdown en-

compasses the interior of spherical voids, then $P_{10} \approx \nu_1 a^3 E_0$, where ν_1 is the linear number of voids which the track of the particle touches upon, a is the radius of the void, and E_0 is the breakdown field.

Let us estimate the power of the incoming radio wave at the angle of maximum intensity. For $L/R \sim 0.1$, $\nu_1 \sim 1$, $a \approx 0.3$ cm, $E_0 \approx 30$ kV/cm and $\omega \sim 3 \times 10^9$ cps we obtain $\mathcal{E}_{\omega} \Delta\omega \approx 10^{-2} c^{-3} \omega^2 \Delta\omega \approx 10^{-12} \Delta\omega$ erg, which for $\Delta\omega \approx 0.1\omega \approx 3 \times 10^9$ and for a signal duration $\tau \sim 1/\Delta\omega \approx 3 \times 10^{-10}$ second corresponds to radiation power $\Delta\omega \approx 1$ W, which is definitely known to be sufficient for the reception of the radiation.

CONCLUSION

The considered effect may find broad application, for example, for the production of bursts of sharply directed radiation and also for the measurement of the speed of flight of a particle based on the pattern of the direction of the radiation, as measured by a system of receivers.

The effect also is of interest for the recording of the frequency of a sequence of fronts of radio or light bursts from spark discharges near the track of the particle in a periodic medium and for an estimate of the speed of a particle based on the frequency of the transient bursts and the angle of reception.

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ON THE THEORY OF THE FERMI LIQUID

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The ground state energy and other characteristics of a Fermi system are calculated using the method of Green's functions and Landau's theory of Fermi liquids. The calculations are carried out for a potential with a long-range and a short-range part such that $n^{1/3}r_s \ll 1$ and $n^{1/3}r_l \gg 1$, where r_s and r_l are the ranges of the short- and long-range parts and n is the density of particles.

INTRODUCTION

LANDAU'S theory of Fermi liquids permits the calculation of the energy of the system, the effective mass of the excitations, the velocity of zero and ordinary sound, and other characteristics of a Fermi liquid. However, the theoretical formulas contain a number of unknown numerical parameters which can not be determined within the framework of a phenomenological theory. These parameters can be calculated only on the basis of a microscopic theory. Up to now, such calculations have only been carried out in the gas approximation, i.e., for the case where the range of the forces is appreciably smaller than the average distance between the particles $n^{-1/3}$, where n is the density of particles.

In the present paper we calculate the vertex part and the basic characteristics of a Fermi system for the case where the interaction potential consists of two terms V_s and V_l , such that V_s can be treated in the gas approximation, i.e., the range r_s of V_s is much smaller than $n^{-1/3}$, whereas V_l satisfies the opposite condition: $r_l \gg n^{-1/3}$, where r_l is the range of V_l . The magnitudes of V_s and V_l can be arbitrarily large. The method of summing over graphs allows one to find the vertex part $\Gamma_{\alpha\beta, \gamma\delta}(p_1, p_2, p_1 + k, p_2 - k)$ for $|k|p_0 > \omega_k$, where p_0 is the Fermi momentum. However, the ground state energy and other characteristics of the system are expressed in terms of the vertex part $\Gamma_{\alpha\beta, \gamma\delta}$ in the limit $p_0|k|\omega_k^{-1} \rightarrow 0$, $\omega_k \rightarrow 0$, i.e., $\Gamma_{\alpha\beta, \gamma\delta}^\omega$. To calculate $\Gamma_{\alpha\beta, \gamma\delta}^\omega$ we use the basic relations of the theory of Fermi liquids.

Fermi liquids with potentials consisting of two terms with $r_s \ll n^{-1/3}$ and $r_l \gg n^{-1/3}$ have already

been studied earlier in their application to the nucleus.^[1,2] Vagrado and Kirzhnits^[1] have taken account of the long-range force V_l with the help of the perspicuous but inaccurate method of the self-consistent field in configuration space. Amus'ya^[2] calculated the energy of the system by the graph technique, but he assumed that the expression for the vertex part of the long-range forces found by summation of graphs is valid for all k and ω . Actually, this expression is correct only when $p_0|k| > \omega$ (see below). Moreover, in^[2] no account was taken of the graph Γ_c , which is of the same order of magnitude as the graphs Γ_a and Γ_b [see formula (7)]. Both in^[1] and^[2] V_l is an attractive potential, so that the normal state of the system is unstable (see below).

THE VERTEX PART FOR $p_0|k| > \omega$

In order to calculate the energy spectrum of the system we must know the vertex part (box diagram) $\Gamma_{\alpha\beta, \gamma\delta}(p_1p_2, p_3p_4)$. Let us assume first that $V = V_s, V_l = 0$. Let us compare the ladder graph of second order (ladder with two rungs, Fig. 1a) with the other nonvanishing graph of second order, viz., the chain with two dotted lines (Fig. 1b). In the first case the matrix element is equal to

$$M_a \sim \int G_0(p_3 - p) G_0(p_4 + p) V(k - p) V(p) d^4p \\ \sim V^2(0)r_s^{-1},$$

$$k = p_2 - p_4 \equiv p_3 - p_1, \quad p = \{p, \omega\},$$

$$G_0(p) = [\omega - \varepsilon_0(p) + \mu + i\delta \operatorname{sgn}(|p| - p_0)]^{-1};$$

μ is the chemical potential, $\varepsilon_0(p) = p^2/2$; $\hbar = m = 1$;

$$p_0 = (3\pi^2 n)^{1/3}.$$

In the second case

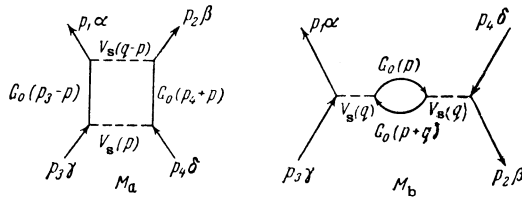


FIG. 1

$$M_b \sim V^2(k) \Pi_0(k),$$

$$\begin{aligned} \Pi_0(k) &= \frac{2}{i(2\pi)^4} \int G_0(p+k) G_0(p) d^4p \\ &= \frac{2}{(2\pi)^3} \int \frac{n_0(p) - n_0(p+k)}{\omega - \varepsilon_0(p+k) + \varepsilon_0(p)} dp; \end{aligned} \quad (1)$$

if $|k| \ll p_0$, then [3]

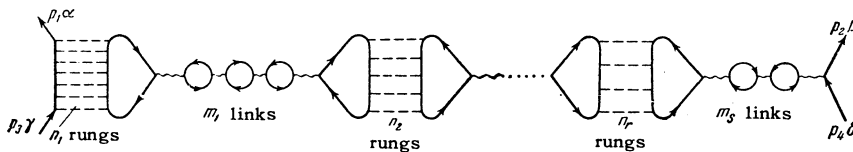
$$\begin{aligned} \Pi_0(k) &\approx \frac{2}{(2\pi)^3} \int \frac{\partial n_0}{\partial \varepsilon} \frac{p_0(nk)}{p_0(nk) - \omega_k} dp \\ &= -\frac{p_0}{\pi^2} \left[1 - \frac{\omega_k}{2|k|p_0} \ln \left| \frac{\omega_k + p_0|k|}{\omega_k - p_0|k|} \right| \right. \\ &\quad \left. + \frac{i\pi|\omega_k|}{2|k|p_0} \theta(p_0|k| - |\omega_k|) \right]; \end{aligned} \quad (2)$$

here $n = p/|p|$, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. $\Pi_0(k) \approx -p_0/\pi^2$ for $p_0|k| \gg \omega_k$ and $\Pi_0(k) \approx p_0^3|k|^2/3\pi^2\omega_k^2$ for $p_0|k| \ll \omega_k$.

Thus $M_b/M_a \lesssim n^{1/3} r_s$. Hence we can restrict ourselves to graphs of the ladder type for $nr_s^3 \ll 1$.

Let us now assume that $V_s = 0$ and $nr_s^3 \gg 1$. Obviously, we can then restrict ourselves to graphs of the chain type if $p_0|k| > \omega$. If, on the other hand, $p_0|k| < \omega$, we must also take into account more complicated graphs, as is seen from (2). If both $V_s \neq 0$ and $V_l \neq 0$ the complete box diagram $\Gamma_{\alpha\beta, \gamma\delta}$ for $p_0|k| > \omega$ can be calculated using only the graphs of the type shown in Fig. 2 and similar graphs in which the external lines for the terminal ladder or chain are both ingoing or both outgoing, and not one ingoing, the other outgoing, as in Fig. 2.

Let us now consider the graph of Fig. 2 and all graphs of the same form which differ from one another only by the number of rungs in any one ladder or the number of links in any one chain. Summing all graphs of this type, we obtain the graph of Fig. 2 in which only one ladder is replaced by the box diagram of the gas approximation,



i.e., by the sum of all graphs of the ladder type $\Gamma_{g\alpha\beta, \gamma\delta}(p_1 p_2, p_3 p_4)$, or one chain replaced by the box diagram $\Gamma_{l\alpha\beta, \gamma\delta}(p_1 p_2, p_3 p_4)$, i.e., the sum of all graphs of the chain type. The expression for Γ_g for a Fermi system has been found by Galitskii: [4]

$$\Gamma_{g\alpha\beta, \gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} \Gamma_g,$$

$$\begin{aligned} \Gamma_g &= f(p', p) + \int dq f(p', q) f^*(p, q) \\ &\times \left[\frac{1}{q^2 - p^2 + i\delta} + \frac{N(q)}{\omega_p - q^2 - P^2/4 + i\delta N(q)} \right], \\ 2p' &= p_1 - p_2; \quad 2p = p_3 - p_4; \quad P = p_1 + p_2; \\ N(q) &= 1 - n(P/2 + q) - n(P/2 - q). \end{aligned} \quad (3)$$

Here $f(p', p)$ is the two-body scattering amplitude multiplied by 4π . For Γ_l we can easily find the equation (see Fig. 3)

$$\Gamma_{l\alpha\beta, \gamma\delta} = \delta_{\alpha\gamma} [\delta_{\beta\delta} V_l(k) + V_l(k) \Pi(k) \Gamma_{l\alpha'\beta, \alpha'\delta/2}],$$

from where we obtain

$$\Gamma_{l\alpha\beta, \gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} \Gamma_l, \quad \Gamma_l = V_l(k) / [1 - V_l(k) \Pi(k)]. \quad (4)$$

The zero order Green's functions $G_0(p)$ in the graphs of Figs. 2 and 3 must, of course, be replaced by the exact Green's functions $G(p)$. Therefore, the expression (3) for G , which has been found using G_0 , is approximate.

The polarization operator $\Pi(k)$, expressed through the exact G , can be calculated in the following way. According to [5, 6], G has close to the pole $\varepsilon = 0$, $|p| = p_0$ the form

$$\begin{aligned} G(p, \varepsilon) &= \frac{a}{\varepsilon - v(|p| - p_0) + i\delta \operatorname{sgn}(|p| - p_0)}, \\ 0 < a &\leq 1, \quad v = \frac{p_0}{m^*}. \end{aligned}$$

The product $G(p) G(p+k)$ under the integral over p can, according to Landau, [6] be written in the form

$$G(p) G(p+k) \approx 2\pi i a^2 \frac{(nk)}{\omega - v(nk)} \delta(\varepsilon) \delta(|p| - p_0) + \varphi(p), \quad (5)$$

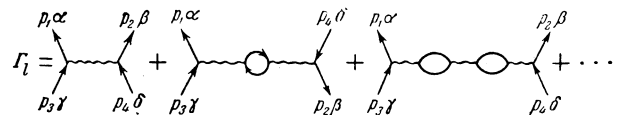


FIG. 3

FIG. 2. Graph with r ladders and s chains. Dotted line: $V_s(p)$, wavy line: $V_l(p)$.

where $\varphi(p)$ is the regular part of $G(p)G(p+k)$. We obtain from (5)

$$\Pi(k) = c - b \left[1 - \frac{\omega}{2v|k|} \ln \left| \frac{\omega + v|k|}{\omega - v|k|} \right| \right] + \frac{i\pi|\omega|}{2v|k|} \theta(v|k| - |\omega|),$$

$$c = \frac{2}{i(2\pi)^4} \int \varphi(p) d^4p, \quad b = \frac{p_0^2 a^2}{v\pi^2}, \quad \lim_{\omega/p_0|k| \rightarrow 0} \Pi = c - b. \quad (6)$$

Extending the summation over graphs of the type of Fig. 2 to the other ladders and chains, we obtain the same graph in which now all ladders are replaced by the box diagram Γ_g and all chains by the box diagram Γ_l . Adding graphs analogous to that of Fig. 2, but with two ingoing or two outgoing lines at all terminal ladders or chains, we obtain for the total box diagram Γ the expression (for $\omega < p_0|k|$):

$$\Gamma = \Gamma_g + \Gamma_l + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4,$$

where $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 are shown in Fig. 4.

Figure 4 leads to a system of four integral equations for $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 , which are given graphically in Fig. 5. This system is easily solved, but in our approximation we can manage without solving it. Indeed, let us estimate the graph Γ_a in Fig. 4 by replacing Γ_g by the constant amplitude f_0 :

$$\Gamma_{\alpha\beta, \gamma\delta}(p_1, p_2; p_3, p_4) = \frac{2i\delta_{\alpha\gamma}\delta_{\beta\delta}}{(2\pi)^4} \int \Gamma_g(p_1, p+k; p_3, p) \times \Gamma_l(p, p_2; p+k, p_4) G(p) G(p+k) d^4p$$

$$\approx 2\delta_{\alpha\gamma}\delta_{\beta\delta} f_0 \Gamma_l(k) \Pi(k);$$

for $\omega \ll p_0|k|$ $\Gamma_l = -2f_0(b-c)\Gamma_l$; $f_0 \sim r_s$.

Thus $\Gamma_a/\Gamma_l \sim n^{1/3}r_s$, i.e., the inclusion of the additional box diagram Γ_g gives the small factor $n^{1/3}r_s$. In calculating Γ_1 and Γ_2 we can therefore neglect all graphs containing two or more box diagrams Γ_g , restricting ourselves to the graphs Γ_a, Γ_b , and Γ_c . Calculating these in the same way as Γ_a , but without the replacement of Γ_g by the constant f_0 , we find [the terms containing φ are not written down in (7)]:

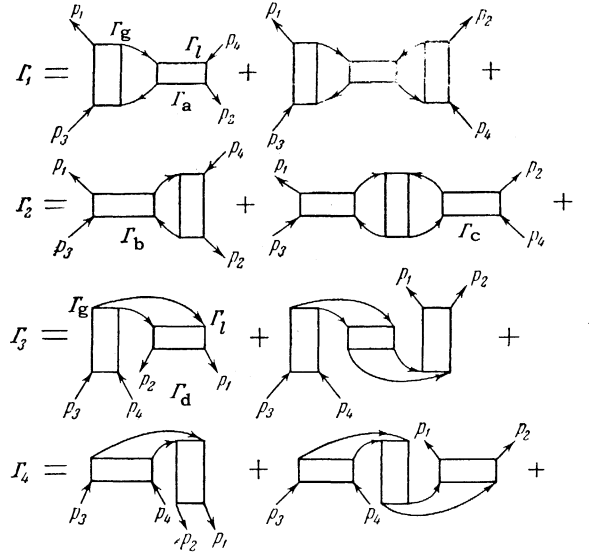


FIG. 4

$$\Gamma_a \approx \frac{(c-b)}{4\pi} \Gamma_l(k) \int \Gamma_g(p_1, \omega_1; p_0n+k, \omega; p_1+k, \omega_1 + \omega; p_0n, 0) \frac{v(kn)}{\omega - v(kn)} d\Omega,$$

$$\Gamma_b \approx \frac{(c-b)}{4\pi} \Gamma_l(k) \int \Gamma_g(p_0n, 0; p_2, \omega_2; p_0n+k, \omega; p_2 - k, \omega_2 - \omega) \frac{v(kn)}{\omega - v(kn)} d\Omega,$$

$$\Gamma_c \approx \frac{(c-b)\Gamma_l(k)}{4\pi} \int \Gamma_b(p_1, \omega_1; p_0n+k, \omega; p_1+k, \omega_1 + \omega; p_0n, 0) \frac{v(kn)}{\omega - v(kn)} d\Omega, \quad (7)$$

if $\Gamma_g = f_0$, then $\Gamma_b = \Gamma_a, \Gamma_c = 4f_0\Gamma_l^2(k)\Pi^2(k)$.

Let us now consider the simplest graph entering in Γ_3 , namely Γ_α :

$$\Gamma_{\alpha\beta, \gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma_\alpha;$$

$$\Gamma_\alpha = \frac{i}{(2\pi)^4} \int \Gamma_g(p, P-p; p_3, p_4) G(P-p) \times G(p) \Gamma_l(p_1, p_2; p, P-p)$$

$$\approx -\frac{a^2}{(2\pi)^3} \int \Gamma_g(p, P-p; p_3, p_4) \Gamma_l(p_1, p_2; p, P-p) \times \frac{dp}{\omega_p - \varepsilon(p) - \varepsilon(P-p) + i\delta \operatorname{sgn}(|p| - p_0)};$$

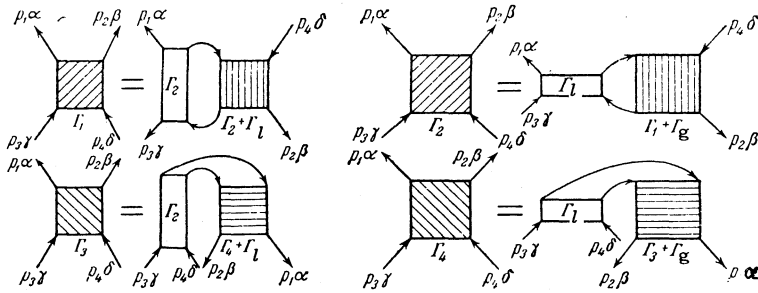


FIG. 5

the integral over $\mathbf{p}$ will converge for such values of $|\mathbf{p}|$ for which the product $\Gamma_g \Gamma_l$ begins to decrease appreciably, i.e., when $|\mathbf{p}| \sim r_l^{-1}$ ($\Gamma_l \rightarrow 0$ for $|\mathbf{p}| \sim r_l^{-1}$ and $\Gamma_g \rightarrow 0$ for $|\mathbf{p}| \rightarrow r_s^{-1}$). Therefore

$$\Gamma_d \approx f_0 \Gamma_l / r_l, \quad \Gamma_d / \Gamma_l \approx r_s / r_l \ll n^{1/3} r_s \quad (8)$$

and hence all graphs entering in Γ_3 and Γ_4 can be neglected.

As far as the long-range component of the potential, V_l , is concerned, there are two possibilities: 1) $U_l r_l^3 n^{1/3} \gtrsim 1$ [U_l is the maximal value of $|V_l(\mathbf{r})|$] and 2) $U_l r_l^3 n^{1/3} \ll 1$. In the second case we can treat the potential V_l in the Born approximation. Here we have to include only the graph of first order in V_l , so that we have for all values of $\mathbf{k}$ and ω

$$\Gamma = \Gamma_g + V_l(\mathbf{k}). \quad (9)$$

In the first case, which is of much greater interest, we have for $p_0 |\mathbf{k}| > \omega$ (see above)

$$\Gamma = \Gamma_g + \Gamma_l + \Gamma_a + \Gamma_b + \Gamma_c, \quad (10)$$

$$\Gamma_g + \Gamma_a + \Gamma_b + \Gamma_c \sim r_s;$$

$$\text{for } U_l r_l^3 n^{1/3} \gg 1, \quad \Gamma_l \approx -\Pi^{-1}(\mathbf{k}) - \Pi^{-2}(\mathbf{k}) V_l^{-1}(\mathbf{k}),$$

$$-\Pi^{-1} \approx (b-c)^{-1} \sim n^{-1/3}, \quad \Pi^{-2} V_l^{-1} \sim n^{-1/3} (U_l r_l^3 n^{1/3})^{-1}.$$

$\Gamma > 0$ for both signs of $V_l(\mathbf{k})$, $\Gamma_g + \Gamma_a + \Gamma_b + \Gamma_c$ depends on p_1 and p_2 , while Γ_l does not.

VERTEX PART FOR ARBITRARY $\mathbf{k}$, ω , AND Γ^ω

Let us first consider the case $U_l r_l^3 n^{1/3} \gtrsim 1$. In

Landau's theory of Fermi liquids the limiting values of Γ : $\Gamma^k = \Gamma$ for $|\mathbf{k}| \rightarrow 0$, $\omega/p_0 |\mathbf{k}| \rightarrow 0$ and $\Gamma^\omega = \Gamma$ for $\omega \rightarrow 0$, $p_0 |\mathbf{k}|/\omega \rightarrow 0$, play an essential role. From the value of Γ obtained by us [formula (10)] we can evidently determine only Γ^k (it is important that in all parts of the graphs entering in Γ_l according to Fig. 3 there are no insertions of the ladder type with $p_0 |\mathbf{k}'| < \omega'$). Let us first write down the symmetrized box diagram $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}$ which is obtained from $\Gamma_{\alpha\beta, \gamma\delta}(p_1 p_2, p_3 p_4)$ by subtracting $\Gamma_{\alpha\beta, \gamma\delta}$ with permuted arguments, $p_3 \gamma \rightleftharpoons p_4 \delta$. If Γ in (10) is regarded as a function of p_1 , p_2 , and $\mathbf{k}$, then

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta} = \Gamma(p_1, p_2, \mathbf{k}) \delta_{\alpha\gamma} \delta_{\beta\delta}$$

$$- \Gamma(p_1, p_2, p_2 - p_1 - \mathbf{k}) \delta_{\alpha\delta} \delta_{\beta\gamma}.$$

From this we find $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^k$:

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^k(p_1, p_2) = \Gamma^k(p_1, p_2, 0) \delta_{\alpha\gamma} \delta_{\beta\delta} - \Gamma(p_1, p_2, p_2 - p_1) \delta_{\alpha\delta} \delta_{\beta\gamma}. \quad (11)$$

In order to obtain $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^\omega$, we use the equation

of Landau's theory of Fermi liquids which connects $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^\omega$ and $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^k$.^[6]

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^\omega(p_1, p_2) = \tilde{\Gamma}_{\alpha\beta, \gamma\delta}^{(k)}(p_1, p_2) + \frac{p_0^2 a^2}{v(2\pi)^3} \int \tilde{\Gamma}_{\eta\beta, \xi\delta}^k(q, p_2) \times \tilde{\Gamma}_{\alpha\xi, \gamma\eta}^\omega(p_1, q) d\Omega_q. \quad (12)$$

The solution of (12) can be found by expanding into a series of Legendre polynomials.^[6] For the zeroth harmonic we must replace $\tilde{\Gamma}_{\eta\beta, \xi\delta}^k$ under the integral in (12) by the average value of $\tilde{\Gamma}_{\eta\beta, \xi\delta}^k$ on the Fermi sphere, $\bar{\Gamma}_{\eta\beta, \xi\delta}^k$:

$$\bar{\Gamma}_{\eta\beta, \xi\delta}^k = \frac{1}{4\pi} \int \tilde{\Gamma}_{\eta\beta, \xi\delta}^k(q, p_2) d\Omega_q.$$

The main term in $\bar{\Gamma}_{\alpha\beta, \gamma\delta}^k$ will be $\Gamma_l^k(p_1, p_2, 0)$:

$$\Gamma_l^k(p_1, p_2, 0) = (b - c + V_l^{-1}(0))^{-1} \sim n^{-1/3},$$

$$\bar{\Gamma}_l^k(p_1, p_2, 0), \bar{\Gamma}_{a,b,c}^k(p_1, p_2, 0), \bar{\Gamma}_g(p_1, p_2, p_2 - p_1) \sim r_s; \quad (13)$$

the average values $\bar{\Gamma}_{a,b,c}(p_1 p_2, p_2 - p_1)$ can be neglected:

$$\bar{\Gamma}_{a,b,c}(p_1, p_2, p_2 - p_1) \sim r_s / n^{2/3} r_l^2;$$

the average value $\bar{\Gamma}_l(p_1 p_2, p_2 - p_1)$ will also be small:

$$\bar{\Gamma}_l(p_1, p_2, p_2 - p_1) \sim n^{-1/3} / n^{2/3} r_l^2. \quad (14)$$

We thus obtain from (12) for the l -th harmonic of $\tilde{\Gamma}^\omega$:

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta}^{\omega l} = [\Gamma^{\omega l} \delta_{\alpha\gamma} \delta_{\beta\delta} - \Gamma'^{\omega l} \delta_{\alpha\delta} \delta_{\beta\gamma}] [1 + b \Gamma'^{\omega l} / 2(2l+1)]^{-1};$$

$$\Gamma^{\omega 0} = \Gamma^{kl} [1 - b(\Gamma^{kl} - \Gamma'^{\omega l} / 2) / 2(2l+1)]^{-1}$$

$$\Gamma^{\omega 0} \approx \frac{\Gamma_l^k}{1 - b(\Gamma_l^k + \Gamma_1^k)}; \quad \Gamma^{\omega l} \approx \Gamma^{kl} \ll \Gamma^{\omega 0} \quad (l \neq 0);$$

$$\Gamma'^{\omega} = \Gamma'_l + \Gamma'_g + \Gamma'_{abc}, \quad (15)$$

where

$$\Gamma_1^k \approx \bar{\Gamma}_g^k / 2 + \Gamma_{abc}^k, \quad \Gamma_{abc} = \Gamma_a + \Gamma_b + \Gamma_c,$$

$$\Gamma_i^k \equiv \Gamma_i^k(p_1, p_2, 0),$$

$$\Gamma'_i \equiv \Gamma_i(p_1, p_2, p_2 - p_1) \quad (i = l, g, a, b, c).$$

To estimate the magnitude of c in (6) and (15), we use the relation for a found in^[3,7]:

$$a^{-1} = 1 - \frac{i}{2} \int \tilde{\Gamma}_{\alpha\beta, \alpha\beta}^\omega(p, q) \varphi(q) d^4 q / (2\pi)^4. \quad (16)$$

We find from (16)

$$a^{-1} \approx 1 + c\Gamma^{\omega_0},$$

and hence

$$-cb^{-2}/(\Gamma_1^k - V_l^{-1}(0)/b^2) \approx 1 - a < 1,$$

$$\Gamma_l^k \approx [b + aV_l^{-1}(0)]^{-1}, \quad \Gamma^{\omega_0} \approx -1/a [b^2\Gamma_1^k - V_l^{-1}(0)]. \quad (17)$$

It follows from (17) that

$$\Gamma^{\omega_0}/\Gamma^k \sim U_l r_l^3 n^{1/3} \quad \text{for } U_l r_l^3 r_s n^{1/3} \ll 1, \\ \Gamma^{\omega_0}/\Gamma^k \sim n^{-1/3} r_s^{-1} \gg 1 \quad \text{for } U_l r_l^3 r_s n^{1/3} \gg 1.$$

Using the basic equation of the theory of Fermi liquids, [6]

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta}(p_1, p_2, k) = \tilde{\Gamma}_{\alpha\beta, \gamma\delta}^{\omega}(p_1, p_2) \\ + \frac{a^2 p_0^2}{(2\pi)^3} \int \tilde{\Gamma}_{\alpha\xi, \gamma\eta}^{\omega}(p_1, q) \tilde{\Gamma}_{\eta\beta, \xi\delta}(q, p_2, k) \frac{v(\mathbf{nk})}{\omega - v(\mathbf{nk})} d\Omega_q; \\ n = \frac{q}{|q|} \quad (18)$$

we can find, by the method of successive approximations, the symmetrized box diagram $\tilde{\Gamma}_{\alpha\beta, \gamma\delta}$ for arbitrary $\mathbf{k}$ and ω :

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta}(p_1, p_2, k) \approx \frac{\Gamma_l^k \delta_{\alpha\gamma} \delta_{\beta\delta}}{1 - b(\Gamma_l^k + \Gamma_1^k)f(k)} \\ - (\Gamma_l' + \Gamma_g' + \Gamma_{abc}') \delta_{\alpha\delta} \delta_{\beta\gamma}, \\ f(k) = \frac{\omega}{2v|\mathbf{k}|} \ln \left| \frac{\omega + v|\mathbf{k}|}{\omega - v|\mathbf{k}|} \right| - \frac{i\pi|\omega|}{2v|\mathbf{k}|} \theta(v|\mathbf{k}| - |\omega|). \quad (19)$$

If we introduce the spin operators σ and σ' , we have

$$\tilde{\Gamma}_{\alpha\beta, \gamma\delta} \approx \frac{\Gamma_l^k}{1 - b(\Gamma_l^k + \Gamma_1^k)f} - \frac{1}{2} (\Gamma_l' + \Gamma_g' + \Gamma_{abc}') \\ - \frac{(\sigma\sigma')}{2} (\Gamma_l' + \Gamma_g' + \Gamma_{abc}'). \quad (20)$$

EFFECTIVE MASS OF THE EXCITATIONS AND ENERGY OF THE SYSTEM

The quantity $a^2 \tilde{\Gamma}_{\alpha\beta, \gamma\delta}^{\omega}$ appears in the expressions for the effective mass of the excitations m^* , the velocity of zero sound (with momentum l) $\tilde{u}_l = s_l v$, the velocity of ordinary sound u , and the ground state energy of the system E_0 . [6]

$$\frac{1}{m^*} - 1 = 1 - \frac{p_0 a^2}{2(2\pi)^3} \int \tilde{\Gamma}_{\alpha\beta, \alpha\beta}^{\omega}(\chi) \cos \chi d\Omega, \quad (21)$$

$$\chi = \angle(\mathbf{p}_1, \mathbf{p}_2),$$

$$\frac{s_0}{2} \ln \left(\frac{s_0 + 1}{s_0 - 1} \right) \approx 1 + 1/\Gamma^{\omega_0} b, \quad (22)$$

$$u^2 = \frac{p_0^2}{3} + \frac{p_0^2 a^2}{6(2\pi)^3} \int \tilde{\Gamma}_{\alpha\beta, \alpha\beta}^{\omega}(\chi) (1 - \cos \chi) d\Omega, \quad (23)$$

$$E_0 = \int \mu(N) dN, \quad \mu(N) = \int u^2(N) \frac{dN}{N}. \quad (24)$$

The difference $1/m^* - 1$ is, according to (21), determined by the first and not the zeroth harmonic of $\tilde{\Gamma}_{\alpha\beta, \alpha\beta}^{\omega}$, i.e., not by Γ^{ω_0} , but by the small quantities $\bar{\Gamma}_l$, $\Gamma_g - \bar{\Gamma}_g$:

$$\frac{1}{m^*} - 1 = 1 - \frac{p_0 a^2}{(2\pi)^3} \int [2(\Gamma_g^k + \Gamma_{abc}^k - \bar{\Gamma}_g^k - \bar{\Gamma}_{abc}^k) \\ - (\Gamma_l' + \Gamma_g' + \Gamma_{abc}')] \cos \chi d\Omega, \\ \frac{1}{m^*} - 1 = \beta_1 + \beta_2, \quad \beta_1 \approx n^{-1/3} r_l^{-2} \quad \beta_2 \approx n^{1/3} r_s^{-2}. \quad (25)$$

In (22) $\Gamma^{\omega_0} b \gg 1$ for $U_l r_l^3 n^{1/3} \gg 1$; therefore [6]

$$s_0 \approx [b\Gamma^{\omega_0}/3]^{1/2}. \quad (26)$$

From (23) we find

$$u^2 \approx \frac{p_0^2}{3} + \frac{p_0^2 a^2}{3\pi^2} \Gamma^{\omega_0} = \frac{p_0^2}{3} - \frac{p_0^2 a}{3\pi^2 [b^2 \Gamma_1^k - V_l^{-1}(0)]}. \quad (27)$$

Substituting (27) in (24), we obtain for $\partial a/\partial n \approx 0$

$$\Delta E_0 = E_0 - E_{\text{ideal gas}} \approx \frac{3aN V_l(0)}{\alpha^2} \left[\frac{3}{4} n^{1/3} - \frac{1}{\alpha} \tan^{-1} \alpha n^{1/3} \right. \\ \left. - \frac{1}{2\alpha^4} \ln(1 + \alpha^2 n^{1/3}) \right],$$

$$\alpha = a^2 3^{1/3} \pi^{-1/3} \sqrt{|\Gamma_l^k| V_l(0)},$$

$$\Delta E_0 \approx - (3\pi^2)^{1/3} N n^{1/3} a^{-3}/4\Gamma_1^k$$

$$\sim E_{\text{ideal gas}}/n^{1/3} r_s \quad \text{for } U_l r_l^3 r_s n^{1/3} \gg 1;$$

$$\Delta E_0 \approx \frac{1}{2} a V_l(0) N n \sim U_l r_l^3 n^{1/3} E_{\text{ideal gas}} \quad \text{for}$$

$$U_l r_l^3 r_s n^{1/3} \ll 1; E_{\text{ideal gas}} = \frac{3}{10} (3\pi^2)^{1/3} N n^{1/3}. \quad (28)$$

To have a system which is thermodynamically stable, i.e., $u^2 = \partial p/\partial \rho > 0$, it is necessary that $\Gamma^{\omega_0} > 0$, i.e.,

$$- \Gamma_1^k b^2 + V_l^{-1}(0) > 0.$$

If there are attractive forces, pairing has to be taken into account.

We see therefore that the correction to the effective mass is small in the presence of long-range forces with $U_l r_l^3 n^{1/3} \gtrsim 1$, just as in the gas model. [4, 8, 9] The ground state energy and the velocity of sound, on the other hand, are appreciably larger than for the ideal gas (by the factor $U_l r_l^3 n^{1/3}$ or $n^{-1/3} r_s^{-1/3}$). This increase is connected with the presence of the long-range forces. It is clear that the transition to the limit $r_l = \infty$, i.e., to Coulomb forces, is not possible in our formulas;

for Coulomb forces the system must be electrically neutral.

In the case where $U_l r_l^3 n^{1/3} \ll 1$, when Γ is given by (9), $\Gamma\omega = \Gamma^k$ and the corrections to m^* , $\tilde{u}$, u , and E_0 are small as compared to the ideal gas values: $\sim n^{1/3} r_s$ or $\sim U_l r_l^3 n^{1/3}$.

In the case of the nucleus the inequalities $n^{1/3} r_s \ll 1$ and $n^{1/3} r_l \gg 1$ do not hold very strongly. Therefore, our model can only be regarded as a limiting case if applied to the nucleus.

In conclusion I express my gratitude to A. I. Larkin for an interesting discussion.

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ON THE THEORY OF DIFFUSION-VISCOUS FLOW OF POLYCRYSTALLINE BODIES

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The mechanism is analyzed and a quantitative theory is developed for the high-temperature diffusion viscous flow of polycrystalline bodies at low stresses. The role of slip along grain boundaries during their motion and diffusion deformation is explained. The viscosity coefficient tensor, which depends on the structural anisotropy of the substance (grain shapes and sizes) and on the magnitude of resistance to slip along grain boundaries, is derived. In addition to the exact value of this tensor, simple interpolation formulas are given which permit investigation of the general flow equations. By way of example the kinetics of the filling-up of a spherical pore under pressure is considered. A study was made of the effects accompanying the process of diffusion-viscous flow: the appearance of nonuniform stress fluctuations in a single grain (stresses of the second kind), variation of the macroscopic elastic moduli during flow, etc. The limits of applicability of the theory are discussed.

1. MECHANISM OF DIFFUSION-VISCOUS FLOW.
STATEMENT OF THE PROBLEM

It is known that, apart from the plastic flow of crystals, which is a threshold effect and occurs under sufficiently high loads, there is also a diffusion-viscous flow in solids at arbitrarily low stresses. The latter process is due to the diffusion of vacancies and is important only at high temperatures when this diffusion is sufficiently intensive. Strictly speaking, this process is dominant only at low stresses when all the other mechanisms of a threshold nature (in particular the formation and motion of dislocations) are inactive.

The basic idea of the mechanism of diffusion-viscous flow was first put forward in the work of Nabarro^[1] and Herring;^[2] the latter work gives calculations and estimates for some typical cases of such flow. However, neither Nabarro nor Herring allowed for some important factors. Consequently there were many obscurities in their picture of the process as a whole and in the conditions for its appearance, their estimates were not always valid. In the present work this mechanism is analyzed from a somewhat different point of view and a derivation is given of a complete system of equations describing the process of diffusion-viscous flow in the general case. At the same time the concomitant phenomena, typical of the mechanism of diffusion-viscous flow, become clear.

The first assumption in the mechanism of diffusion-viscous flow is that the surfaces bounding

structural units in a real crystalline body (separate grains in a polycrystal or separate blocks in a coarse-mosaic single crystal) are virtual sources and sinks for vacancies. In a perfect crystalline lattice a vacancy cannot be generated or annihilated by itself but a sufficiently defective surface represents, as it were, an "amorphous" layer which is capable of reversible rearrangement when some vacancies reach it: it becomes as it were "compressed" to its initial density and vacancies disappear.¹⁾

On the basis of this assumption the mechanism of flow is as follows: in a body subjected to a macroscopically uniform but anisotropic stress σ_{ik} , various normal stresses $\sigma_{ik}n_i n_k$ (n is the vector normal to the surface) act on various portions of crystallite surfaces. Consequently near these surfaces the extent of saturation with vacancies varies: the chemical potential μ of a vacancy at the surface (which is a virtual source of vacancies) will, by virtue of the equilibrium between the annihilation and generation of vacancies, be equal to

$$\mu = \mu_0 + \sigma_{nn}\omega_0, \quad \mu - \mu_0 = kT \ln(c/c_0) \quad (1)$$

(c_0 is the equilibrium concentration of vacancies at $p = 0$; μ_0 is the chemical potential for the concentration c_0 ; ω_0 is the volume per unit vacancy);

¹⁾An analysis of the conditions under which a block boundary can be considered as a surface with virtual sources is given in Appendix 1.

hence the value of the vacancy excess $\Delta c = c - c_0$ is determined by the relationship

$$\Delta c = c_0 \omega_0 \sigma_{nn} / kT. \quad (2)$$

The resultant concentration gradients within a grain give rise to diffusion currents of vacancies (or equivalent currents of atoms in the opposite direction). These currents produce a change of the grain shape without a change of its volume. The characteristic feature of this process is the absence of macroscopic diffusion currents in the body as a whole and the presence only of complex microscopically nonuniform currents within each grain.

Deformation of the whole crystalline body consisting of a very large number of grains is a self-consistent process in the sense that discontinuities are not produced at the grain boundaries and the diffusion-viscous deformation of a particular grain depends on the deformations of the grains which surround it. The analysis given below shows that this necessarily produces stress fluctuations within each grain (stresses of the second kind) in spite of a macroscopically uniform stress state of the whole sample.

The self-consistent change in the shape of the grains is the result of the directional motion of their centers of gravity which produces macroscopic changes in the shape of the body (flow). This displacement of the centers of gravity of the grains in diffusion-viscous flow is similar to the displacement of molecules in the process of the viscous flow of liquids, and is necessarily accompanied by slip along grain boundaries.

If we introduce an average value of the stress tensor $p_{ik} = \bar{\sigma}_{ik}$ in a small macroscopic element of volume containing a large number of grains, and also an average value of the velocities of displacement of particles in this element $V_i = \bar{v}_i$, then a macroscopic description of the process of flow is equivalent to the establishment of a relationship between the tensor p_{ik} and the rate-of-deformation tensor $V_{ik} = \frac{1}{2} (\partial V_i / \partial x_k + \partial V_k / \partial x_i)$. In the general case of quasi-steady-state flow such a relationship should have the form

$$p_{ik} = p_0 \delta_{ik} + \alpha_{iklm} V_{lm}, \quad V_{ii} = 0, \quad (3)$$

where the viscosity tensor α_{iklm} is governed by the sizes and shapes of the grains, i.e., by the "structural anisotropy" of the polycrystal.

The equation of continuity $V_{ii} = 0$ characterizes the aforementioned constancy of the grain volume and the absence of discontinuities. In the simplest case of structural isotropy α_{iklm} reduces to a scalar η and Eq. (3) becomes

$$p_{ik} = -p \delta_{ik} + \eta V_{ik}, \quad -p = p_{ii}/3. \quad (3a)$$

According to Eq. (2) the variation of the vacancy concentration δc on the grain surface is governed by the non-spherical (shear) part $p'_{ik} = p_{ik} + p \delta_{ik}$ of the stress tensor: $\delta c = c_0 \omega_0 p' / kT$. Therefore, the rate of deformation related to the diffusion currents $j_i = D_0 \partial c / \partial x_i$ should be of the order of magnitude of

$$V_{ik} \sim D_0 \partial^2 c / \partial x_i \partial x_k \sim D_0 \delta c / L^2 \sim p' D_0 c_0 \omega_0 / L^2 kT$$

(L is the grain size). This gives an estimate of the expected magnitude of the viscosity tensor α_{iklm} :

$$\alpha_{iklm} \sim \eta \sim kTL^2 / D\omega_0, \quad D = D_0 c_0$$

(D_0 is the vacancy diffusion coefficient and D is the atomic self-diffusion coefficient). We shall see, however, that this estimate is valid only in the case of free slip along grain boundaries.

The macroscopic coordinate dependence of the tensors p_{ik} and V_{ik} is determined by the equilibrium equation

$$\partial p_{ik} / \partial x_k = 0, \quad (4)$$

which, together with Eq. (3), forms a complete system for the determination of the four quantities V_i and p_0 :

$$\frac{\partial p_0}{\partial x_i} + \frac{\partial}{\partial x_k} \alpha_{iklm} \frac{\partial V_l}{\partial x_m} = 0, \quad \text{div } \mathbf{V} = 0. \quad (4a)$$

Thus our problem reduces to the derivation of Eq. (3) and the determination of α_{iklm} in terms of the structural anisotropy of the polycrystal and the properties of the grain boundaries.

2. SELF-CONSISTENT DISPLACEMENT AND DEFORMATION OF GRAINS. ALIGNMENT OF STRESSES. ROLE OF SLIP ALONG GRAIN BOUNDARIES.

We shall consider a macroscopically uniform portion of the polycrystal (large compared with the grain dimensions) with given constant values of the average quantities p_{ik} and V_{ik} ($V_{ii} = 0$). We shall select a microscopic element of volume (small compared with the grain dimensions) and follow its motion. The mass-current density, or in other words the average velocity $\mathbf{v}$ of this element, consists of a part $\mathbf{v}^\alpha$, related to the motion of the grain as a whole, and a part related to the vacancy-diffusion current $\mathbf{j} = -D_0 \nabla c$; since the current of atoms is equal to the current of vacancies in the opposite direction, then ²⁾

²⁾Hereafter we shall assume that the diffusion coefficient is isotropic.

$$\mathbf{v} = \mathbf{v}^\alpha + D_0 \nabla c. \quad (5)$$

From the continuity condition it follows that at the boundary of two grains denoted by α and β the normal components of the velocity v_n should be equal. According to Eq. (5) this gives the expression

$$D_0 \frac{\partial c}{\partial n} \Big|_\alpha - D_0 \frac{\partial c}{\partial n} \Big|_\beta = v_n^\beta - v_n^\alpha, \quad (6)$$

which represents the conditions for self-consistent motion of the grains and diffusion currents.

Since the concentration of vacancies is very small, only a negligibly small deformation ($\sim c_0$) can take place during the time necessary to establish a quasi-steady-state diffusion current, and consequently during the whole process the distribution of vacancies is given by $\Delta c = 0$ (according to Eq. (5) this means $\text{div } \mathbf{v} = 0$).

For convenience in later treatment, instead of the concentration of vacancies we shall use a potential proportional to it

$$\varphi = \mu - \mu_0/\omega_0 = (kT/c_0\omega_0)(c - c_0). \quad (7)$$

As mentioned above, the potential φ is governed by the equation

$$\Delta\varphi = 0 \quad (8)$$

and by the boundary conditions at the surfaces of separation of the grains

$$\delta \left(\frac{\partial \varphi}{\partial n} \right) \Big|_S = \frac{\partial \varphi}{\partial n} \Big|_\alpha - \frac{\partial \varphi}{\partial n} \Big|_\beta = \kappa (v_n^\beta - v_n^\alpha), \quad \kappa = \frac{kT}{D\omega_0}. \quad (9)$$

Moreover, according to the equilibrium condition of Eq. (1) the value of φ at each point on a grain boundary represents the normal stress at that surface

$$\varphi|_S = \sigma_{nn}. \quad (10)$$

Strictly speaking, the equilibrium condition of Eq. (10) in the presence of currents $\approx \nabla \varphi$ should be replaced by the dynamic condition $\varphi|_S - \sigma_{nn} = \lambda \partial \varphi / \partial n$. However, if the natural assumption is made that $\lambda \approx a$ (a is the lattice constant), the term $\lambda \partial \varphi / \partial n$ may be neglected.

An important conclusion follows from the conditions formulated above: since the conditions of Eqs. (9) and (10) cannot be satisfied independently, then for the quasi-steady-state flow to appear, a preliminary alignment of the stresses near a grain boundary should occur, making Eqs. (9) and (10) compatible. This can easily be explained by a simple example: let us consider the diffusion-viscous compression of a single-crystal cube under the pressure p of a perfectly hard piston (Fig. 1). Obviously the upper face of the cube can be dis-

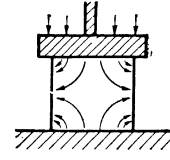


FIG. 1

placed only as a whole. Initially the normal stress σ_{nn} at all points of this face is constant and equal to $-p$; consequently $\varphi = -p$ at the upper face and $\varphi = 0$ at the lateral faces. This means that very strong diffusion currents appear near the corners and weak diffusion currents in the center. Since the rate of displacement of the boundary cannot match these currents at each point, elastic deformations appear on the surface due to the redistribution of atoms when the boundary position is fixed.

Then σ_{nn} is no longer constant in the boundary and only $\bar{\sigma}_{nn} = -p$; in particular the pressure decreases near the corners, becoming zero at the corner. Steady-state motion of the piston is achieved when the pressure distribution over the surface is such that the diffusion current is constant at all points on the upper face.³⁾

Thus the initial elastically uniform state of the body is destroyed in the process of establishing diffusion-viscous flow and is replaced by a complex distribution of stresses inside each grain (stresses of the second kind). It is necessary to emphasize particularly that the stresses of the second kind appearing in diffusion-viscous flow are not identical to the stress fluctuations in polycrystals due to the elastic anisotropy^[3] of the separate crystallites, and may even occur in the isotropic model. The actual distribution of these stresses will be considered below.

A second important point in the characteristics of the stressed state is related to the role of slip along "amorphous" layers at grain boundaries. It follows from Eq. (10) that the potential φ and consequently also its tangential derivative are continuous at the grain boundary. This means, according to Eq. (5), that the tangential components of the velocity v_t have a discontinuity at the boundary $\delta v_t|_S = v_t^\beta - v_t^\alpha$ (here $\delta v_t|_S$ is the velocity of slip along the boundary). Thus the self-consistent change of the shape of blocks and their displacement in the process of diffusion-viscous flow are necessarily accompanied by slip along boundaries. (Obviously for an arbitrary grain shape the vanishing values of all $v_t^\beta - v_t^\alpha$

³⁾An analysis of the process of stress "alignment" and of the concomitant phenomena will be given in a separate communication.

mean that for all grains $v^\alpha = 0$ and nothing can occur except a directional diffusion current in the body as a whole.)

An estimate of the tangential stresses σ_{tn} at the boundary due to flow can be obtained from the following considerations: we shall represent the viscosity of the "amorphous" layer at the boundary by a coefficient η_S and relate it to the resistance to slip by the self-evident equality

$$\sigma_{tn}|_S = \eta_S (v_t^\beta - v_t^\alpha)/a \quad (11)$$

(a is the lattice constant). Obviously $v_t^\beta - v_t^\alpha \approx LV_{ik} \approx L\delta\sigma_{nn}/\eta^0 a$, $\eta^0 \approx \kappa L^2$ (L is the grain size, $\delta\sigma_{nn}$ is the variation of σ_{nn} along the boundary) consequently

$$\sigma_{tn} \sim \eta_S \delta\sigma_{nn}/\kappa a L, \quad \kappa = kT/D\omega_0. \quad (11a)$$

The viscosity of the amorphous layer is clearly independent of the grain size and for a sufficiently defective boundary (at high temperatures) its order of magnitude is $\eta_S \approx \kappa a^2 \approx kT/Da$. This gives us an estimate $\sigma_{tn}|_S \approx a\delta\sigma_{nn}/L$, i.e., in practice $\sigma_{tn}|_S = 0$. Thus in this case the distribution of stresses is given by the condition (10) for the normal stresses, and by zero values of the tangential stresses at the surfaces of separation between the grains.

However, if the surface of separation between the grains is insufficiently amorphous, then slip may be greatly impeded and the estimate $\eta_S \approx \kappa a^2$ is no longer valid. If η_S is so large that $\eta_S L/a\eta^0 \gg 1$, then according to Eq. (11a) the tangential stresses are very large:

$$\sigma_{th} \sim \eta_S LV_{ik}/a \gg \eta^0 V_{ik}.$$

In this case the rate of deformation is limited by slip. A simple estimate of the non-spherical part of p'_{ik} gives

$$p'_{ik} = \overline{\sigma'_{ik}} \sim \sigma_{tn} \sim \eta^* V_{ik}, \quad \eta^* = \eta_S L/a,$$

i.e., the effective viscosity of the polycrystal η^* is considerably greater than the expected value $\eta = \eta^0$. The estimate (11a) shows that in this case the normal stresses are practically constant at the boundary

$$\delta\sigma_{nn} \sim \frac{\eta_0}{\eta_S} \frac{a}{L} \sigma_{tn} \sim \frac{\eta_0}{\eta_S} \frac{a}{L} p' \ll p'.$$

Therefore, the diffusion currents are much smaller than the estimate in the preceding case and "align" themselves automatically with the deformation rate governed by the possibility of slip.

3. SOLUTION OF THE DIFFUSION PROBLEM. CALCULATION OF THE VISCOSITY TENSOR

Let us return now to the diffusion problem. Equation (8) with the condition (9) corresponds to the electrostatic problem of the potential distribution in the presence of a network of surface charges with density $4\pi q = -\kappa (v_n^\beta - v_n^\alpha)$. Its solution has the well-known form

$$\varphi(r) = \varphi_0 + \int \frac{q dS}{R'}, \quad R' = R - r,$$

where integration is carried out over the whole network of surfaces of separation (R is the integration variable, and φ_0 is constant because of the macroscopic uniformity and is determined below).

The basis of the derivation (given below) of a relationship between V_{ik} and p_{ik} is that, by selecting fixed values of V_{ik} , we can find the average stresses in the medium during flow. Consequently before proceeding further it is necessary to relate the velocities of motion of individual grains to the macroscopic deformation-rate tensor $\hat{V} = V_{ik}$.

When the deformation is completely uniform the velocity at the point R is $V = \hat{V}R$, where $V_i = V_{ik}X_k$, taking account of the rotation of the medium as a whole; consequently if R^α is the coordinate of the center of gravity of the α -th grain, then

$$v_i^\alpha = V_{ik} X_k^\alpha + u_i^\alpha, \quad (12)$$

where u^α are the random deviations of the velocity of motion of the α -th grain from the average value of the velocity of matter at this point ($\bar{u} = 0$).

Taking into account the fact that vector normals to the surface of each of neighboring grains have different signs at the surface and using Eq. (12), we transform the expressions for q^α into the form

$$\begin{aligned} q^{\alpha\beta} &\equiv q^\alpha + q^\beta = -(\kappa/4\pi) (v^\alpha n^\alpha + v^\beta n^\beta) \\ &= -(\kappa/4\pi) [(v^\alpha - \hat{V}R) n^\alpha + (v^\beta - \hat{V}R) n^\beta], \\ 4\pi q^\alpha &= \kappa V_{ik} \xi_k^\alpha n_i - \kappa u_i^\alpha n_i^\alpha, \quad \xi^\alpha = R - R^\alpha. \end{aligned} \quad (13)$$

Here $\xi^\alpha = R - R^\alpha$ are the coordinates of a point on the surface of separation, reckoned from the center of gravity of the corresponding grain.

Therefore

$$\begin{aligned} \int \frac{q dS}{R'} &= \sum_\alpha \int \frac{q^\alpha dS^\alpha}{R'_\alpha} = \frac{\kappa}{4\pi} \sum_\alpha \left\{ V_{ik} \int \frac{n_i \xi_k^\alpha dS^\alpha}{R'_\alpha} - \int \frac{u_i^\alpha n_i^\alpha dS^\alpha}{R'_\alpha} \right\} \\ &= \frac{\kappa}{4\pi} V_{ik} \sum_\alpha \int_{\Omega_\alpha} \frac{\partial}{\partial \xi_i} \left(\frac{\xi_k}{R'_\alpha} \right) d\xi - \frac{\kappa}{4\pi} \sum_\alpha \int \frac{u_i^\alpha n_i^\alpha dS^\alpha}{R'_\alpha}, \\ R'_\alpha &= R^\alpha + \xi - r, \end{aligned} \quad (14)$$

where dS_α is an element of area on the surface of the α -th grain, and Ω_α is the volume of this grain.

For an arbitrary scatter of the shape and dimensions of the grains, the random deviations of the grain velocities u^α are, in general, not equal to zero and should be determined from the requirement that the total force and momentum acting on each grain separately should vanish.⁴⁾ In order to eliminate the quantity u^α , we shall consider first a portion of the medium consisting of equivalent grains (cf. the plane model in Fig. 2). In spite of some elements of artificiality in this model, it allows us to introduce a description of the structural anisotropy in terms which are related to the shape and dimensions of the grains and the nature of the grain packing. Obviously, because of the equivalence of the grains, there are no random deviations of the velocity ($u^\alpha = 0$).

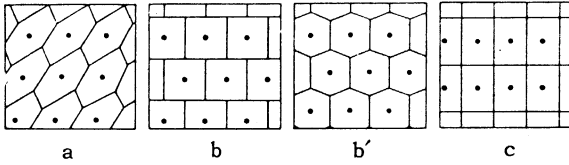


FIG. 2

This means that, in particular, the discontinuity $\partial\varphi/\partial n$ at the boundary between the grains α and β has the simple form

$$\delta(\partial\varphi/\partial n)|_S = V_{ik} R_k^{\alpha\beta} n_i; \quad (15)$$

$R^{\alpha\beta} = R^\alpha - R^\beta$ is the distance between the centers of gravity of the neighboring grains. Consequently, according to Eq. (14), we have the following exact formula for the potential:

$$\begin{aligned} \varphi(\mathbf{r}) &= \varphi_0 + \kappa V_{ik} \psi_{ik}(\mathbf{r}) \quad (V_{ii} = 0), \\ \psi_{ik}(\mathbf{r}) &= \frac{1}{4\pi} \sum_\alpha \int \xi_k \frac{\partial}{\partial \xi_i} \left(\frac{1}{R_\alpha} \right) d\xi. \end{aligned} \quad (16)$$

Integration in Eq. (16) is carried out over the volume of one grain. If the origin of coordinates $\mathbf{r} = 0$ is taken at the center of gravity, then R_α represents the vector in a lattice based on centers of gravity of the grains.

To obtain the required relationship (3) we have to express the magnitude of the average stresses in a grain p_{ik} in terms of the potential φ and the

deformation velocity V_{ik} , taking into account the equalities (10) and (11). It is known that the average value $p_{ik} = \overline{\sigma_{ik}}$ is expressed in terms of surface forces f_i by the formula

$$p_{ik} = \frac{1}{\Omega_0} \int x_i f_k dS.$$

The normal component, $f_n = \sigma_{nn}|_S$, and the tangential component, $f_t = \sigma_{tn}|_S$, of the force are given by Eqs. (10) and (11), and hence

$$\begin{aligned} f_k &= \varphi n_k + g_k \eta_S / a, \quad g_k = V_{kl} R_l^\beta - n_k n_l V_{lm} R_m^\beta, \\ p_{ik} &= p_0 \delta_{ik} + \frac{1}{\Omega_0} \int x_i \frac{\partial \varphi}{\partial x_k} d\mathbf{r} + \frac{\eta_S}{a \Omega_0} \int x_i g_k dS, \quad p_0 = \frac{1}{\Omega_0} \int \varphi d\mathbf{r} \end{aligned}$$

(Ω_0 is the grain volume, and R^β is the distance between the centers of gravity of adjoining grains).

Finally, allowing for Eq. (16), we obtain

$$\begin{aligned} p_{ik} &= p_0 \delta_{ik} + \alpha_{iklm} V_{lm}, \quad \alpha_{iklm} = \alpha_{iklm}^0 + \alpha_{iklm}^*, \\ \alpha_{iklm}^0 &= \frac{\kappa}{\Omega_0} \int x_i \frac{\partial \psi_{lm}}{\partial x_k} d\mathbf{r}, \\ \alpha_{iklm}^* &= \frac{\eta_S}{a \Omega_0} \int (x_i R_l^\beta \delta_{km} - x_i n_k n_l R_m^\beta) dS. \end{aligned} \quad (17)$$

Thus the viscosity tensor consists of two components, one of which represents diffusional viscosity in the case of zero resistance to slip, and the second is entirely due to this resistance. In order of magnitude

$$\alpha_{iklm}^0 \sim \eta^0 \sim \kappa L^2, \quad \alpha_{iklm}^* \sim \eta^* \sim \eta_S L / a.$$

We have seen that in the case of easy slip (amorphous surface of separation) $\eta_S \approx \kappa a^2$ and $\eta^* / \eta^0 \approx a/L \ll 1$; then $\alpha_{iklm} = \alpha_{iklm}^0$. In the case of strong resistance to slip $\eta^* / \eta^0 \gg 1$ and $\alpha_{iklm} = \alpha_{iklm}^*$.

The expression for α_{iklm}^* can be simplified by an approximation (averaging process) consisting of replacing the vector R^β , which connects the centers of gravity of adjoining grains to the point $\mathbf{r}$, by double the distance to the point of contact: $R^\beta(\mathbf{r}) \approx 2\mathbf{r}$. This gives⁵⁾

$$\alpha_{iklm}^* = \frac{2\eta_S}{a \Omega_0} \int (x_i x_l \delta_{km} - x_i x_m n_k n_l) dS. \quad (17a)$$

Thus, for a spherical grain of radius R the tensor α_{iklm}^* reduces to the scalar $\eta^* = 2\eta_S R / 5\pi a$.

Let us now analyze the diffusion component α_{iklm}^0 . In the present work we shall consider mainly the case of easy slip (reasonably defective boundary) and, therefore, subsequently, unless

⁴⁾Each quantity u^α is specified by two constant vectors u_0^α and ω^α : $u^\alpha = u_0^\alpha + [\omega^\alpha \times \xi^\alpha]$, representing the relative velocity of the center of gravity of the grain and the angular velocity of the grain. Thus the number of conditions for mechanical equilibrium is the same as the number of unknowns.

⁵⁾Here and subsequently the quantity α_{iklm} should be made symmetrical for each pair of the indices i, k and l, m , if this symmetry is lost in the approximate formulas (17a), (18a), etc.

otherwise specified, we shall assume $\alpha_{iklm} = \alpha_{iklm}^0$. It follows from Eqs. (16) and (17) that in this case

$$\alpha_{iklm} = \frac{\kappa}{4\pi\Omega_0} \sum_{\alpha} \int_{\Omega_0} \int_{\Omega_0} x_i \xi_m \frac{\partial^2}{\partial x_k \partial \xi_l} \left(\frac{1}{R_{\alpha}} \right) dx d\xi; \quad (18)$$

$$R_{\alpha} = R^{\alpha} + \xi - \mathbf{x}.$$

It should be noted that since the coordinates $\mathbf{x}$ and ξ in the integral (18) are reckoned from the center of gravity of a grain ($\int \Omega_0 \mathbf{x} d\mathbf{x} = \int \Omega_0 \xi d\xi = 0$), the terms of the series (18) decrease as R^{-5} and the series converges very rapidly. Therefore, in the majority of cases, satisfactory accuracy is obtained by taking from the exact formula (18) one term of the series with R^{α} (cf. footnote ⁵):

$$\alpha_{iklm} = \frac{\kappa}{4\pi\Omega_0} \int_{\Omega_0} \int_{\Omega_0} x_i \xi_m \frac{\partial^2}{\partial x_k \partial \xi_l} \frac{1}{|\mathbf{x} - \xi|} dx d\xi. \quad (18a)$$

Transforming to the nondimensional coordinates $x_i \rightarrow x_i \Omega_0^{1/3}$, $\xi_i \rightarrow \xi_i \Omega_0^{1/3}$, we obtain

$$\alpha_{iklm} = \eta \gamma_{iklm}, \quad \eta = kT \Omega_0^{1/3} / D \omega_0,$$

$$\gamma_{iklm} = \frac{1}{4\pi} \int_{\omega} \int_{\omega} x_i \xi_m \frac{\partial^2}{\partial x_k \partial \xi_l} \frac{1}{|\mathbf{x} - \xi|} dx d\xi, \quad (19)$$

where the integrals γ_{iklm} are taken over a non-dimensional portion ω of a grain of unit volume and represent numbers which depend only on the grain shape.

An interpolation formula, which makes it possible to generalize the expression for the viscosity tensor to the case of an arbitrary distribution of grain shapes and dimensions, is obtained by averaging the expression (18a) over this distribution. The meaning of such averaging may be obtained from two points of view. On the one hand, such averaging is exact for a packing model with grains varying in size and shape slowly from point to point (the changes being random). On the other hand, in the exact formula obtained from Eq. (14) for the general case, averaging involves neglecting the grain velocity fluctuations u^{α} . Thus, for example, if we represent the grain distribution by the probability $W(\mathbf{x}, \xi) d\mathbf{x} d\xi$ that two points, separated by the distances $\mathbf{x}$ and ξ from the center of gravity of the grain, belong to this grain, then the tensor α_{iklm} can be written in the form

$$\alpha_{iklm} = \frac{\kappa}{4\pi\Omega} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\mathbf{x}, \xi) x_i \xi_m \frac{\partial^2}{\partial x_k \partial \xi_l} \frac{1}{|\mathbf{x} - \xi|} d\mathbf{x} d\xi,$$

$$\bar{\Omega} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\mathbf{x}, \xi) d\mathbf{x} d\xi. \quad (20)$$

It should be noted that the tensor α_{iklm} is specified, correct to terms of the type $A_{ik} \delta_{lm}$

and $\delta_{ik} B_{lm}$, since $A_{ik} V_{ll}$ disappears because of the condition $V_{ll} = 0$ and $\delta_{ik} (B_{lm} V_{lm})$ can be taken back to the first term in Eq. (3). Therefore Eq. (18a) can be written in another form:

$$\alpha_{iklm} = \frac{\kappa}{4\pi\Omega_0} \int_{S_0} \int_{S_0} \frac{x_i x'_m dS_k dS'_l}{|\mathbf{x} - \mathbf{x}'|}, \quad (18b)$$

where the integral is taken over the grain surface, and $dS_k = n_k dS$.

A convenient method of investigating the anisotropy of α_{iklm} is based on the Fourier representation of the integrand in Eq. (18a). Simple transformations, using the equality

$$\frac{1}{r} = \frac{1}{2\pi^2} \int e^{i\mathbf{k}\mathbf{r}} \frac{d\mathbf{k}}{k^2},$$

give ⁶)

$$\alpha_{iklm} = \frac{\kappa}{8\pi^3\Omega_0} \int_{-\infty}^{\infty} k_l k_k \frac{\partial Q}{\partial k_i} \frac{\partial Q^*}{\partial k_m} \frac{d\mathbf{k}}{k^2},$$

$$Q = \int_{\Omega_0} e^{i\mathbf{k}\mathbf{x}} d\mathbf{x}. \quad (18c)$$

Thus, for spherical grains, Q depends only on the modulus of $\mathbf{k}$ and the tensor α_{iklm} , being isotropic, reduces to the scalar of Eq. (3a).

For very strong grain-shape anisotropy, for example in the case when the grains are greatly elongated along one direction ($C/A \gg 1$, where C is the length of the grain and A represents its transverse dimensions), it is insufficient to use in Eq. (18) only the term with $R^{\alpha} = 0$; it is necessary also to retain the terms with $R^{\alpha} \approx C$ in the transverse direction.

On the basis of the equality $V_{ll} = 0$ it can be easily shown that in the case of axial symmetry there are only three independent components of α_{iklm} . If x_3 is the selected axis, then

$$\rho_{\alpha\beta} = \rho_0 \delta_{\alpha\beta} + \eta_1 V_{\alpha\beta}, \quad \alpha, \beta = 1, 2,$$

$$\rho_{33} = \rho_0 + \eta_3 V_{33}; \quad \rho_{\alpha 3} = \eta_2 V_{\alpha 3}, \quad -\rho = \rho_0 + \frac{1}{3} (\eta_3 - \eta_1) V_{33}. \quad (21)$$

A simple analysis of the expressions in Eq. (18) shows that when $C \gg A$ the orders of magnitude are given by $\eta_1 \sim \kappa A^2$, $\eta_2 \sim \kappa AC$, $\eta_3 \sim \kappa C^2$.

4. EXAMPLE: THE CASE OF A PARALLELEPIPEDAL STRUCTURE

In spite of the relatively compact form of Eqs. (18a)–(18c) for the tensor α_{iklm} , these formulas

⁶)It should be noted that a similar representation of the exact formula (18) reduces to a substitution of integration over the space $\mathbf{k}$ in Eq. (18c) with summation over the points $\mathbf{H}$, where $\mathbf{H}$ are vectors of the reciprocal lattice with respect to R^{α} .

are far too complicated for practical applications. Therefore, we shall consider here one case which can be solved exactly, giving the simplest expressions for the viscosity tensor. We refer to the rectangular parallelepiped grain shape and packing of the type shown in Fig. 2c.

If the edges of the parallelepiped are of lengths $2A_1, 2A_2, 2A_3$, then, according to Eq. (15), the discontinuity of the derivative $\partial\varphi/\partial n$ at the grain boundaries is given by

$$\delta\partial\varphi/\partial n|_{x_i=A_i(1+2N)} = 2\kappa V_{ii}A_i^2, \quad i = 1, 2, 3 \quad (22)$$

where N is an integer.

It is easily seen that the periodic harmonic function satisfying the conditions of Eq. (22) (in each grain the coordinates x_i are reckoned from its center of gravity) is

$$\varphi(\mathbf{r}) = \varphi_0 + \frac{1}{2}\kappa(V_{11}x_1^2 + V_{22}x_2^2 + V_{33}x_3^2), \quad V_{ii} = 0. \quad (23)$$

For p_{ik} the formula (17) gives

$$p_{ik} = p_0\delta_{ik} + \frac{1}{3}\kappa\delta_{ik}V_{kk}A_k^2; \\ -p = p_{ii}/3 = p_0 + \theta, \quad \theta = \frac{\kappa}{3}\sum_k V_{kk}A_k^2, \quad p_0 = \varphi_0 + \theta/2, \quad (24)$$

and, therefore, the nonspherical part of the stress tensor $p'_{ik} = p_{ik} + p\delta_{ik}$ is

$$p'_{ii} = p_{ii} + p = -\frac{1}{2}\kappa\theta + \frac{1}{3}\kappa V_{ii}A_i^2, \quad i = 1, 2, 3, \\ V_{ii} = p'_{ii} + \sum_k p'_{kk}A_k^2 / \sum_k A_k^2. \quad (24a)$$

The relationships in Eq. (24) express the relationship being sought between the stresses and the deformation rate V_{ik} . We shall consider especially the case $A_1 = A_2 = A$, $A_3 = C$:

$$p_{33} = p_0 + \frac{\kappa}{3}V_{33}C^2, \quad p_{11} + p_{22} = 2p_0 - \frac{\kappa}{3}V_{33}C^2.$$

Then

$$p'_{33} = p_{33} + p = \frac{\kappa}{3}V_{33}\frac{2C^2 + A^2}{3}, \\ p'_{11} + p'_{22} = -p'_{33} = -\frac{\kappa}{3}V_{33}\frac{2C^2 + A^2}{3}. \quad (24b)$$

For $C = A$ all the axes are equivalent and $p'_{ii} = (1/3)\kappa A^2 V_{ii}$ ($i = 1, 2, 3$). Finally, in the planar case ($V_{33} = 0$, $V_{11} = -V_{22}$) $p_{11} = p_0 + (1/3)\kappa A^2 V_{11}$, $p_{22} = p_0 - (1/3)\kappa A^2 V_{11}$, or

$$p_{11} + p = -p_{22} - p = \frac{1}{3}\kappa V_{11}(A^2 + B^2), \\ V_{11} = \frac{3}{2}(p_{11} - p_{22})/\kappa(A^2 + B^2). \quad (24c)$$

It is clear from Eq. (24) that in the case considered the nondiagonal components p_{ik} vanish. This is related to some degeneracy, due to the model of grain packing which has been assumed:

on this model, corresponding to Fig. 2c, the points of contact between the grains are degenerate (the general nondegenerate case of contact is shown in Figs. 2a and 2b). A simple analysis shows that in this case there is instability with respect to any nondiagonal component of p_{ik} . Figure 3 shows, by way of example, the deformations for this model under the action of forces applied along the diagonals of the squares. Consequently the model gives reasonable results only in the case when the load conditions are such that the nondiagonal components of p_{ik} are absent.

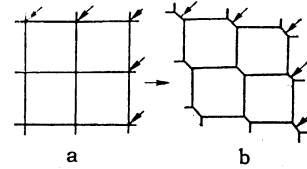


FIG. 3

Another characteristic of this model is related to the fact that the diagonal components of the deformation velocities V_{11}, V_{22}, V_{33} produce no slip along grain boundaries, while the nondiagonal components produce slip only, without the need for diffusion currents. When forced slip is allowed, the formulas (24a)–(24c) for the diagonal components of p_{ik} remain unaltered, while for the nondiagonal components the general formula (17) gives a simple equality $p_{ik} = \eta_S(A_i + A_k)V_{ik}/a$, $i \neq k$.

We shall now deal with the problem of the stress distribution in the case considered. According to Eq. (10) the stresses are strongly inhomogeneous at the boundary:

$$\sigma_{nn}|_{x_i=\pm A_i} = \varphi_0 + \frac{1}{2}\kappa A_i^2 V_{11} + \frac{1}{2}\kappa(x_2^2 V_{22} + x_3^2 V_{33}). \quad (25)$$

For simplicity we shall consider only the planar case ($V_{33} = 0$). Expressing φ_0 and $V_{11} = -V_{22}$ in Eq. (25) in terms of average stresses p_{11} and p_{22} , we shall write the boundary conditions in the form

$$\sigma_{xx}|_{x=\pm A} = -p + (B^2/3 - y^2)M, \\ \sigma_{yy}|_{y=\pm B} = -p + (x^2 - A^2/3)M, \\ \sigma_{xy}|_{x=\pm A} = \sigma_{xy}|_{y=\pm B} = 0, \quad M = \frac{3}{4}(p_{11} - p_{22})/(A^2 + B^2). \quad (26)$$

From the theory developed above it is clear that the results obtained so far are completely unrelated to the elastic properties of the medium; in particular the distribution of stresses at a grain boundary, given by Eqs. (25) and (26), is independent of these properties. However, in general, the stress distribution in the interior of a grain

depends strongly on the elastic moduli. In our example the elastic moduli disappear from the final formulas only in the case of an elastically isotropic medium. In that case it can easily be shown that σ_{xx} , σ_{yy} and σ_{xy} have, over the whole volume of the grain, the same distribution as at the boundary:

$$\begin{aligned}\sigma_{xx} &= p_{11} + \frac{p_{11} - p_{22}}{4} \frac{1 - 3(x/A)^2}{1 + (B/A)^2}, \\ \sigma_{yy} &= p_{22} + \frac{p_{22} - p_{11}}{4} \frac{1 - 3(y/B)^2}{1 + (A/B)^2}, \\ \sigma_{xy} &= 0, \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})\end{aligned}\quad (27)$$

where ν is Poisson's ratio.

This result is naturally affected by the presence of elastic anisotropy. It should be noted that the appearance of inhomogeneous stresses as a result of alignment and the distribution of these stresses should be one of the more important criteria in an experimental verification of the theory of the flow process given here. Obviously in the limiting case of a single crystal the equivalent boundary conditions can be modeled by a system of rigid pistons at the faces of the crystal.

5. ELASTIC ENERGY AND MACROSCOPIC ELASTIC MODULI OF A POLYCRYSTAL IN THE PROCESS OF DIFFUSION-VISCOUS FLOW

We shall consider the macroscopic elastic properties of a polycrystal during the process of diffusion-viscous flow.

The elastic energy of a macroscopic element of volume consists of two parts: E_0 , related to the average stresses p_{ik} , and a fluctuation part E^* , related to the deviations $\sigma_{ik}^* = \sigma_{ik} - p_{ik}$ from these average values. In particular if the crystallites forming the system are elastically isotropic, the elastic energy per macroscopic unit of volume is

$$\begin{aligned}E &= E_0 + E^*, \\ E_0 &= p^2/2K + p_{ik}'^2/4\mu, \quad p_{ik}' = p_{ik} + p\delta_{ik}, \\ E^* &= \sigma_{ik}^{*2}/4\mu, \quad \sigma_{ik}^* = \sigma_{ik} - p_{ik}\end{aligned}\quad (28)$$

where K is the hydrostatic bulk modulus and μ is the shear modulus. As shown above, σ_{ik}^* is linearly related to V_{ik} and consequently to the nonspherical part of the stress tensor p_{ik}' . Thus the fluctuation part of the elastic energy has the form

$$E^* = E^*(p_{ik}) = \nu_{iklm} p_{ik}' p_{lm}' / 4\mu, \quad (29)$$

where the numerical values of ν_{iklm} are determined by the structural anisotropy of the crystal.

Thus, for example, in the planar flow considered at the end of the preceding section this energy is

$$E^* = \frac{1 + \nu^2}{5\mu} \left(\frac{p_{11} - p_{22}}{4} \right)^2$$

where ν is Poisson's ratio.

Thus the elastic energy for given stresses p_{ik} acquires an additional term during flow which is of the same order as the main part of the energy related to uniform homogeneous stresses. (The expression "of the same order" implies the absence of a small physical parameter, although the numerical value of the ratio of the energies may be quite small for grains of near-spherical shape.)

The relationship between the macroscopic (average) values of the elastic deformation ϵ_{ik} and the stresses p_{lm} is obtained by differentiating the elastic energy E with respect to the stresses p_{ik} :

$$\begin{aligned}\epsilon_{ik} &= c_{iklm} p_{lm}, \quad c_{iklm} = c_{iklm}^0 + c_{iklm}^*, \\ c_{iklm}^0 &= \frac{\partial^2 E_0}{\partial p_{ik} \partial p_{lm}}, \quad c_{iklm}^* = \frac{\partial^2 E^*}{\partial p_{ik} \partial p_{lm}}.\end{aligned}\quad (30)$$

The above equation shows that the tensor c_{iklm}^* is anisotropic. This means that during flow the anisotropy of the macroscopic moduli changes and even a medium consisting initially of elastically isotropic crystallites becomes, in general, elastically anisotropic.

There is another important point as follows. Because of the positive quadratic form of $E^*(p_{ik})$ the eigenvalues of the matrix c_{iklm} , arranged in increasing order, are larger than the eigenvalues of the matrix c_{iklm}^0 . The reciprocals of these eigenvalues are the basic elastic moduli of the polycrystal. This means that as a result of diffusional flow the elastic moduli of a polycrystal become smaller. For an elastically isotropic medium this means a reduction of the Lamé coefficients.

6. VARIATION OF THE GRAIN SHAPE AND OF THE VISCOSITY TENSOR DURING FLOW

The flow changes the grain shape and the structural anisotropy. This alters the viscosity tensor α_{iklm} . A microscopic deformation of a single grain is described by the velocity of displacement of the points $\mathbf{x}^S$ on its boundary due to the diffusion current:

$$\frac{d\mathbf{x}^S}{dt} = \frac{1}{\kappa} \nabla \varphi|_S,$$

where φ is given by Eq. (16). However, qualitatively or "on the average," we may assume that

the grain shape varies as if it were deformed uniformly together with the whole medium. This means that

$$dx_i^S/dt = V_{ik}x_k^S. \quad (31)$$

Correspondingly the derivative $d\alpha_{iklm}/dt$ is determined by varying α_{iklm} with the shape of the surface. Using the general formula for the change in an integral when the integration region is varied

$$\delta \int_{\Omega_0} \Psi d\mathbf{x} = \int_S \Psi \delta x_i^S dS_i,$$

we can easily obtain the detailed microscopic formulas by means of equalities (18) and (31). These formulas are, however, complicated and of little interest, and it is more helpful to use the approximate approach developed below.

If the final deformation of the medium at a time t is represented by the matrix $\hat{\lambda} = \lambda_{ik}$, in the sense that any small vector with initial coordinates x_i^0 transforms into the vector $x_i = \lambda_{ik}x_k^0$, it follows from Eq. (31) that

$$d\hat{\lambda}/dt = \hat{V}\hat{\lambda}. \quad (32)$$

In view of the condition $V_{ii} = 0$ (conservation of volume) the determinant $D(\lambda_{ik}) = 1$. V_{ik} is taken at the point of location of the grain at the time t and, in general, depends on time and coordinates. Initially $\lambda_{ik}(0) = 1$. If the "average" shape of the grains is initially spherical [$(x^0x^0) = 1$], then the value of $\lambda_{ik}(t)$ is given by the ellipsoid $(\hat{\lambda}^{-1}\mathbf{x}, \hat{\lambda}^{-1}\mathbf{x}) = 1$ with semiaxes λ_i (λ_i are the principal values of $\hat{\lambda}$; $\lambda_1\lambda_2\lambda_3 = 1$). Therefore, it is natural to represent the structural anisotropy by the ellipsoid λ_{ik} . If the initial anisotropy is given by the matrix $\hat{\lambda}^0$, and the deformation at time t is $\hat{\lambda}(t)$, then the anisotropy at the time t is $\hat{\lambda}(t)\hat{\lambda}^0$ [obviously $D(\hat{\lambda}\hat{\lambda}^0) = D(\hat{\lambda})D(\hat{\lambda}^0) = 1$]. Thus, for example, if $\hat{V}$ is independent of the coordinates and of time, we obtain from Eq. (32)

$$\hat{\lambda} = e^{\hat{V}t}\hat{\lambda}^0.$$

If, on the contrary, $\hat{V} = \hat{V}(s)$ depends on the point s on the trajectory of the grain, and $v(s) = ds/dt$ is the grain velocity, then $d\hat{\lambda}/ds = \hat{V}(s)\hat{\lambda}/v(s)$. If $\hat{V}(s)$ remains diagonal along the trajectory, then

$$\frac{d\lambda_i}{ds} = V_{ii}(s)\lambda_i/v(s), \quad \lambda_i(s) = \exp \int_{s_0}^s V_{ii}(s) \frac{ds}{v(s)}.$$

To obtain a complete solution of the problem of flow it remains to select a reasonable approximation for α_{iklm} in a medium with structural anisotropy given by the ellipsoid λ_{ik} . For this purpose

we can use the formulas (18) and (18a), where the region of integration Ω_0 is the ellipsoid $\hat{\lambda}$ or the approximate estimates given in formulas of the type of (21) or (24). It follows from these formulas that good interpolation is obtained on the assumption ⁷⁾

$$p_{ik} = p_0\delta_{ik} + \eta\lambda_i\lambda_kV_{ik}, \quad \lambda_1\lambda_2\lambda_3 = 1, \quad (33)$$

or, for the nonspherical part,

$$p'_{ik} = p_{ik} - p_{ii}\delta_{ik}/3 = p_{ik} + p\delta_{ik},$$

$$p'_{ik} = \eta\lambda_i\lambda_kV_{ik} - \eta\delta_{ik}\sum_l \lambda_l^2 V_{ll}/3,$$

$$\eta V_{ik} = \lambda_i^{-1}\lambda_k^{-1}p'_{ik} - \delta_{ik}\lambda_i^{-2}\sum_l \lambda_l^{-2}p'_{ll}/\sum_l \lambda_l^{-2}, \quad (33a)$$

where $\eta \approx \kappa\Omega_0^{2/3} = kT\Omega_0^{2/3}/Dc_0\omega_0$ is the isotropic viscosity when $\lambda_i = 1$.

Thus, for example, for the axially symmetric structure $\lambda_3 = \lambda$, $\lambda_1 = \lambda_2 = \lambda^{-1/2}$, taking into account the equality $V_{11} + V_{12} = -V_{33}$ we obtain

$$\begin{aligned} p'_{33} &= p_{33} + p = \frac{1}{3}(2\lambda^2 + \lambda^{-1})V_{33}, & p'_{\alpha 3} &= p_{\alpha 3} = \eta\lambda^{1/2}V_{\alpha 3}, \\ p'_{\alpha\beta} &= p_{\alpha\beta} + p\delta_{\alpha\beta} = \frac{1}{3}\eta(\lambda^2 - \lambda^{-1})V_{33} + \eta\lambda^{-1}V_{\alpha\beta}, \\ & \alpha, \beta = 1, 2. \end{aligned} \quad (34)$$

If we use the relationships (33) and (32), then the equilibrium conditions of Eq. (4) lead to a nonlinear system

$$\begin{aligned} \frac{\partial p_0}{\partial x_i} + \eta \frac{\partial}{\partial x_k} \lambda_{il}\lambda_{km} \frac{\partial V_{lm}}{\partial x_m} \mathbf{V} &= 0, \quad \text{div } \mathbf{V} = 0, \\ \frac{d\lambda_{ik}}{dt} &= V_{il}\lambda_{ik}. \end{aligned} \quad (35)$$

As an example of such a calculation, the filling-up of an isolated spherical pore under the action of hydrostatic pressure is considered in Appendix 2.

The whole analysis in this section is based on the assumption that the diffusion-viscous mechanism is the only cause of the displacement of grain boundaries and the change of their shape. However, the presence of inhomogeneous stresses inside the grains and the related fluctuation energy E^* increase the probability of processes of boundary displacement by recrystallization. Such processes may be particularly important in the case of strong structural anisotropy (for example, greatly elongated grains), when the fluctuation energy and the

⁷⁾In the invariant matrix form the relationships (33) and (33a) have the form

$$\begin{aligned} d &= p_0 + \eta\hat{\lambda}\hat{V}\hat{\lambda}, \quad \hat{p}' = \eta\hat{\lambda}\hat{V}\hat{\lambda} - \eta/3(\text{Sp}\hat{\lambda}^2\hat{V}), \\ \eta\hat{V} &= \hat{\lambda}^{-1}\hat{p}'\hat{\lambda}^{-1} - \hat{\lambda}^{-2}(\text{Sp}\hat{\lambda}^{-2}\hat{p}'/\text{Sp}\hat{\lambda}^{-2}). \end{aligned}$$

excess surface energy are relatively large. Since, however, the characteristic velocities of the process of recrystallization are governed by quite different parameters, there is one region where practically the only process is the diffusion-viscous flow, and another where the change in the shape and dimensions of the grains is governed by recrystallization processes. In the latter region the variation of the viscosity tensor α_{iklm} with time is naturally independent of that in the former region, although the basic equations (3) and (4a) are retained.

In some cases the diffusion of vacancies along grain boundaries, where the diffusion coefficient D_S is considerably greater, may be important in the mechanism of diffusion-viscous flow. This type of vacancy diffusion is a small effect if $D_S a/L \approx D_S (\omega_0/\Omega_0)^{1/3} \ll D$ (a is the atomic spacing and L represents the grain dimensions). In the opposite limit when $D_S a/L \gg D$ the whole process is governed by surface and not by volume diffusion. This obviously may occur in sufficiently small grains. Finally, on increase of p_{ik} the low-threshold processes of diffusion-viscous flow (for example, the generation and motion of dislocations) may be important. An analysis of some of these processes will be given in a separate paper.

APPENDIX 1

BOUNDARY CONDITIONS ON THE SURFACES OF BLOCKS IN A MOSAIC CRYSTAL

An analysis of the conditions under which mosaic block boundaries are capable of absorbing and emitting vacancies has been given by Herring [2], but not all the arguments and results of his analysis seem to be justified. Here we shall investigate the behavior of such boundaries from another point of view.

A boundary between blocks with a low tilt angle θ can be considered as an array of dislocations lying in the plane of this boundary separated by a mean distance d , so that $\theta \approx a/d$, where a is the atomic spacing. The absorption or emission of Δn vacancies, uniformly distributed between dislocations in an element of area of the boundary (dimensions $\Delta L \gg d$), produces motion of the dislocation network to the right or left, and this involves performing an amount of work $\pm \sigma_{nn} \omega_0 \Delta n$. Therefore, if we consider uniform "consistent" average motion of dislocations along the boundary then in such motion each dislocation represents a virtual source or sink of vacancies.

From this point of view the absence of complete freedom of arbitrary displacement of a single dis-

location when the other dislocations are fixed is not very important and does not alter the condition (2) for equilibrium vacancy concentration at a single dislocation.

In the electrostatic analogy each dislocation is represented by a wire of radius a (a is the atomic spacing) at a potential $\varphi_0 = \sigma_{nn}$. The problem consequently reduces first to the question of under what conditions a grid of such wires, separated by a distance d from each another, can be considered as a uniform plane with a potential $\varphi = \varphi_0$. For this to be so it is obviously necessary for the potential dip between the wires, $\delta\varphi \approx e \ln(d/a)$, where e is the "charge," to be considerably smaller than the change in the potential along the surface $\delta\varphi \approx p'$ (p' is the nonspherical part of the stress tensor). The average density of "charge" on the surface (i.e., the discontinuity $\partial\varphi/\partial n$) is obviously $q^* \approx p'/L$ (L is the grain dimension), i.e., the charge is $e \approx q^* d \approx p'd/L$. Thus the condition $\delta\varphi \ll \delta\varphi_0 \approx p'$ gives

$$(d/L) \ln(d/a) \ll 1 \text{ or } \theta/\ln\theta \gg a/L.$$

However, much more important is the second limitation. As shown at the end of Sec. 2, for the flow mechanism described to be realized it is necessary to have the possibility of slip along the grain boundaries. The estimate of the tangential stresses at the boundary, given in Sec. 2, leads to an additional condition for the effective viscosity η_S of the boundary layer (the surface of separation between blocks): $\sigma_{tn}/p' \approx \eta_S/\kappa aL \ll 1$. For sufficiently large tilt angles θ a reasonable estimate of η_S is $\eta_S \approx \kappa d^2$. This gives $d^2/aL \ll 1$, and hence we have a restriction on the magnitude of the tilt angle: $\theta^2 \gg a/L$. However, if θ is sufficiently small then the crystal between dislocations is in general perfect, and diffusional slip along the boundary is excluded practically completely. Therefore, for sufficiently small θ the mechanism described could apply only in those exceptional cases when, due to the symmetry of the block structure and the applied stress, flow occurs without slip at the boundaries (for example as in the case of rectangular parallelepipeds along the direction of the applied forces, discussed in Sec. 4).

APPENDIX 2

KINETICS OF THE FILLING-UP OF A SPHERICAL PORE UNDER PRESSURE

The problem of the kinetics of the filling-up of an isolated spherical pore in a polycrystalline body at various stages of this process has been

considered in [4]. Here we shall consider the influence of the change in the grain shape on this process.

Initially an infinite sample contains a spherical pore of radius $R_0 \gg L$, where L is the dimension of the grains. An external hydrostatic pressure p_0 produces a diffusion-viscous flow toward the center of the pore. The equation of equilibrium in spherical coordinates has the form

$$\frac{1}{r^3} \frac{\partial}{\partial r} r^3 (p_{rr} + p) + \frac{\partial p}{\partial r} = 0 \quad (p = -p_{ii}/3). \quad (2.1)$$

From the equation of continuity it follows that

$$r^2 V_r = R^2 \dot{R}, \quad (2.2)$$

where $R = R(t)$ is the radius of the pore at a time t . Hence

$$V_{rr} = \frac{\partial V_r}{\partial r} = -\frac{2V_r}{r} = -\frac{R^3}{r^3} V_{RR},$$

$$V_{RR} = V_{rr}|_{r=R} = -\frac{2\dot{R}}{R}. \quad (2.3)$$

From the symmetry of the problem it is clear that the structural anisotropy due to flow has, at each point, axial symmetry connected to the elongation of the grains in the radial direction. According to Eq. (35) this leads to a relationship between $(p_{rr} + p)$ and V_{rr} :

$$p_{rr} + p = \eta(\lambda) V_{rr}, \quad \eta(\lambda) = \frac{1}{3} \eta (2\lambda^2 + 1/\lambda), \quad (2.4)$$

$$V_{RR} \frac{R^3}{r^3} \frac{\partial \eta(\lambda)}{\partial r} + \frac{\partial p}{\partial r} = 0. \quad (2.5)$$

The grain deformation parameter λ (the major semiaxis of the deformation ellipsoid) and also the pressure are functions of r and t [or of r and $R(t)$].

The boundary conditions for Eq. (2.5) (taking into account that $p_{rr}|_{r=k} = 2\gamma/R$, where γ is the surface tension) are

$$p|_{r=\infty} = p_0, \quad p(R) + \frac{\gamma}{R} = \eta(\lambda_R) V_{RR},$$

$$\lambda|_{r=\infty} = 1, \quad \lambda|_{r=R} = \lambda_R. \quad (2.6)$$

From Eqs. (2.3), (2.5) and (2.6) we find

$$p(R) - p_0 = -\int_{\infty}^R \frac{R^3}{r^3} \frac{\partial \eta(\lambda)}{\partial r} dr \cdot V_{RR} = \eta(\lambda_R) V_{RR}$$

$$- p_0 - \frac{\gamma}{R}$$

or

$$p_0 + \frac{\gamma}{R} = V_{RR} \left\{ \eta(\lambda_R) + \int_{\infty}^R \frac{R^3}{r^3} \eta'(\lambda) \frac{d\lambda}{dr} dr \right\}$$

$$= \eta V_{RR} \left\{ 1 + \frac{1}{3} \int_1^{\lambda_R} \left(1 + \frac{R^3}{r^3} \right) \left(4\lambda - \frac{1}{\lambda^2} \right) d\lambda \right\}.$$

If a grain with an initial coordinate r_0 is at r

$= r(t, r_0)$ at a time t , it follows from the conservation of volume that $\lambda = (r_0/r)^2$; on the other hand from Eq. (2.2) we have $r^3 - R^3 = r_0^3 - R_0^3$.

Eliminating r_0 from these two equations we obtain a relationship between r , R and λ :

$$\frac{R^3}{r^3} = \frac{\lambda^{3/2} - 1}{\lambda_R^{3/2} - 1}, \quad \lambda_R = (R_0/R)^2.$$

Hence

$$2\eta \dot{R}/R = (p_0 + \gamma/R) / [1 + f(R_0/R)],$$

$$f(z) = \frac{2}{3} \int_1^z \left[1 + \frac{x^3 - 1}{z^3 - 1} \right] \left(4x^2 - \frac{1}{x^4} \right) x dx.$$

In the initial stage $R \approx R_0$, and $f(1) = 0$, and we obtain the formula given in [4]: $p_0 + \gamma/R = -2\eta \dot{R}/R$. The velocity $\dot{R}/R$ then decreases due to grain growth in the radial direction. However, due to fluctuations of the initial shape and dimensions of the grains the formulas cease to be valid when the transverse grain dimensions at the boundary of the pore become of the order of the scatter of the initial dimensions, i.e., when $\lambda_R \approx (L/\delta L)^2$. The axial symmetry in the neighborhood of the pore is then lost and the transverse dimensions of the grains at the pore boundary cease to decrease: the grains "drift" on top of each other until the whole pore is found within the limits of one or two grains. According to [4] this gives a dependence of $\dot{R}/R$ on R of the type shown in Fig. 4.

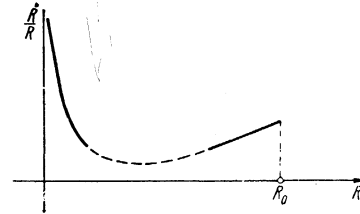


FIG. 4

APPENDIX 3

THE CASE OF INTENSE SURFACE DIFFUSION ALONG GRAIN BOUNDARIES

If the process of surface diffusion along grain boundaries is intense, it may govern the mechanism of diffusion-viscous flow. Variation of $\varphi = \sigma_{nn}$ along the grain boundary gives rise to a volume diffusion current, as well as surface currents

$$j_s = (D_S \omega_0 a / kT) \nabla_s \varphi$$

where D_S is the surface self-diffusion coefficient referring to a monatomic layer, $\nabla_s \varphi$ is the sur-

face gradient of the potential φ , and a is the atomic spacing. On allowing for these currents the conditions (6) and (9), which relate the velocity of the relative motion of the grains $\delta v_n = v_n^\alpha - v_n^\beta$ to the diffusional inflow of additional matter to a given part of the boundary, are altered: together with the volume term

$$\left(\frac{\partial \varphi}{\partial n} \Big|_\alpha - \frac{\partial \varphi}{\partial n} \Big|_\beta \right) \frac{1}{\kappa}$$

there are sources on the surface due to the surface diffusion:

$$q_S = \text{div } \mathbf{j}_S = \frac{a}{\kappa_S} \Delta_S \varphi, \quad \kappa_S = kT/D_S \omega_0.$$

Thus the condition replacing Eq. (9) becomes

$$\kappa (v_n^\beta - v_n^\alpha) = \frac{\partial \varphi}{\partial n} \Big|_\alpha - \frac{\partial \varphi}{\partial n} \Big|_\beta + \frac{\kappa a}{\kappa_S} \Delta_S \varphi, \quad \frac{\kappa}{\kappa_S} = \frac{D_S}{D} \quad (3.1)$$

If the surface self-diffusion coefficient is so large that $D_S/D \gg L/a$, then surface diffusion plays the dominant role and the main contribution to the right-hand part of Eq. (3.1) is given by the term $\approx \Delta_S \varphi$, i.e.,

$$\Delta_S \varphi = \kappa_S (v_n^\beta - v_n^\alpha)/a. \quad (3.2)$$

Equation (3.2) replaces Eqs. (8) and (9) and its solution determines the quasi-steady-state distribution of the normal stresses $\sigma_{nn} = \varphi$; the tangential stresses are, as before, governed by the resistance to slip and are expressed by the conditions (11).

The order of magnitude of the variation of σ_{nn} along the grain surface follows from Eq. (3.2):

$$\delta \sigma_{nn} \sim L^2 \kappa_S (v_n^\alpha - v_n^\beta)/a \sim L^3 \kappa_S V_{ik}/a.$$

For free slip the nonspherical part of the stress tensor is

$$p' \sim \delta \sigma_{nn} \sim L^3 \kappa_S V_{ik}/a \sim \eta_0 V_{ik} LD/a D_S,$$

i.e., the effective viscosity η becomes

$$\eta \sim L^3 \kappa_S/a = \eta_0 LD/a D_S \quad (LD/a D_S \ll 1). \quad (3.3)$$

We shall give concrete expressions for the tensor α_{iklm} in the case of surface diffusion only

for the parallelepiped packing model considered in Sec. 2, para. 4. In that case Eq. (3.2) becomes

$$\Delta_S \varphi|_{x_i=\pm A_i} = -2\kappa_S \frac{A_i}{a} V_{ii}, \quad i = 1, 2, 3.$$

It can easily be shown that the solution of the above equations, continuous at all lines of contact (parallelepiped edges), can be represented by values of the function

$$\varphi = \varphi_0 + \frac{\kappa_S}{2} \sum_i \Lambda_i x_i^2, \quad \Lambda_i = 2 \frac{A_i V_{ii}}{a} - \sum_l \frac{A_l V_{ll}}{a},$$

on the surfaces $x_i = \pm A_i$:

$$\sigma_{nn} = \varphi_S = \varphi|_{x_i=\pm A_i}.$$

Consequently, as before [Eqs. (17), (23), (24)], we obtain the relationship sought between p_{ik} and V_{ik} :

$$p_{ik} = p_0 \delta_{ik} + \frac{\kappa_S}{3} \delta_{ik} \Lambda_i A_i^2,$$

i.e.,

$$p_{ii} = p_0 + \frac{\kappa_S}{3a} \left\{ 2A_i^3 V_{ii} - \sum_l A_l^2 A_l V_{ll} \right\}.$$

In the case of cubic packing ($A_1 = A_2 = A_3 = A$):

$$p_{ii} = p_0 + \frac{2}{3} \kappa_S \frac{A^3}{a} V_{ii}.$$

In the planar case ($V_{33} = 0$):

$$p_{22} - p_{11} = \frac{\kappa_S}{3} \frac{A_1 + A_2}{a} (A_1^2 + A_2^2) V_{11}.$$

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THEORY OF ELECTRICAL CONDUCTIVITY OF IONIC SEMICONDUCTORS IN STRONG ELECTRIC AND MAGNETIC FIELDS

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The electrical conductivity of ionic semiconductors is investigated in strong crossed electric and magnetic fields. The case is considered in which the scattering of the electron energy is due to piezoelectric or optical phonons and momentum scattering is due to ionized impurities. The mean electron energy increases with increase in the electric field strength, and the frequency of electron collisions with ionized impurities decreases. As a result, the value of the conductivity current decreases with growth of the field in a certain electric field intensity range.

AS is well known, the scattering of conduction electrons in semiconductors at temperatures below the Debye temperature is almost elastic. Landau and Kompaneets^[1] and Davydov^[2] have shown that, as a consequence, the electrons slowly transfer to the lattice the energy gained from the external electric field E , and a heating of the electron gas takes place. For a sufficiently large value of the field, the mean energy of the electrons $\bar{\epsilon}$ and their mean free path l become especially dependent on the value of E , which leads to departures from Ohm's law.

In the works mentioned, the interaction with acoustic phonons was taken into account. Davydov and Shmushkevich^[3] considered the case of an ionic semiconductor when the scattering of electrons was brought about by their interaction with optical phonons. The spectrum of optical vibrations of the cubic lattice has the form

$$\omega(q) = \omega_0 (1 - a^2 q^2), \quad (1)$$

where $\omega(q)$ is the frequency of a phonon with wave vector q , ω_0 is the limiting frequency, and a is a length of the order of the lattice constant.

If the mean energy of electrons $\bar{\epsilon}$ is small in comparison with the limiting energy of optical phonons $\hbar\omega_0$, then most of the electrons do not give the energy extracted from the electric field directly to the lattice. Only the electrons with energies ϵ exceeding the energy of the phonons $\hbar\omega$ can emit optical phonons. The process of energy relaxation of the electron gas in this case is that the electrons with $\epsilon \gtrsim \hbar\omega_0$, on the average, emit phonons of higher frequency than the frequency of phonons absorbed by electrons with energies $\epsilon \ll \hbar\omega_0$.

The theory of galvanomagnetic phenomena in strong electric fields in the classic case $\hbar\Omega \ll T$ was constructed by Bass;^[4] here $\Omega = eH/mc$; m is the effective mass of the electron; e is its electric charge; H is the magnetic field intensity, c the velocity of light, and T the lattice temperature in energy units.

A paper by the authors^[5] was devoted to the quantum theory of nonlinear galvanomagnetic phenomena in semiconductors for the case $\hbar\Omega \gg T$. We assumed that the scattering of the energy of the electrons was brought about by interaction with acoustic phonons, while the momentum scattering was brought about by phonons and impurities. It was shown that in the case $\hbar\Omega \gg T$, for $E \perp H$, the dependence of the current on the electric field has an S-shaped form.

The present communication is devoted to a study of the electrical conductivity of ionic semiconductors in strong crossed electric and magnetic fields. We restrict ourselves to a consideration of the low-temperature case, in which the energy relaxation of the electrons is brought about by piezoelectric phonons or by weak dispersion of optical phonons. We consider here an electric field for which the energy of the electrons is much smaller than the limiting energy of the optical phonons $\hbar\omega_0$. Furthermore, we assume that the scattering of electrons is brought about primarily by ionized impurities, but in this case we neglect collisions of electrons with one another. This is valid in the case of a compensated impurity semiconductor, in which the concentration of ionized impurities is large in comparison with the concentration of conduction electrons. Moreover, we shall assume that all impuri-

ties are ionized and that the electron concentration n does not depend on the value of the electric field.

1. In the case of crossed fields $\mathbf{E} \perp \mathbf{H}$, the conductivity current and the generation of Joule power are brought about by the scattering of the electrons. Upon scattering, the center of the electronic orbit is displaced by a quantity of the order of the Larmor radius R . As a consequence, a change in energy of the electron of order eER takes place. If this quantity exceeds the energy transferred by the electrons to the lattice in the time between two such collisions, then the electron gas is heated and its distribution function differs strongly from equilibrium. For small inelasticity of the scattering, the energy relaxation time of the electrons τ_ϵ is large in comparison with the momentum relaxation time $\tau_p: \tau_p \ll \tau_\epsilon$. Therefore, the part of the electron distribution function F_0 that is even in the momentum $\mathbf{p}$ quickly becomes symmetric about the direction of $\mathbf{p}$ and is a function only of the energy ϵ . The part of the distribution function $f(\mathbf{p}) = -f(-\mathbf{p})$ that is odd in the electron momentum, corresponding to the conduction current, is expressed in terms of the symmetric part $F_0(\epsilon)$ in the same fashion as in the theory that is linear in E .

The symmetric part of the distribution function $F_0(\epsilon)$ is found from the condition for the stationary mode, which represents the equation of continuity in energy space. It means that the power of the electric field absorbed by electrons with energy ϵ is equal to the power given to the lattice by the same electrons. It was shown in [5] that in the case of interaction of the electrons with longitudinal acoustic phonons and impurities, the equation for $F_0(\epsilon)$, for $\epsilon \gg \hbar\Omega \gg \hbar(\nu_i + \nu_{ph})$, has the form

$$F_0(\epsilon) + T \left\{ 1 + \frac{1}{3} \left(\frac{cE}{sH} \right)^2 [1 + \nu_i(\epsilon)/\nu_{ph}(\epsilon)] \right\} \frac{\partial F_0}{\partial \epsilon} = 0. \quad (2)$$

Here $\nu_i(\epsilon)$ and $\nu_{ph}(\epsilon)$ are the collision frequencies of electrons with energy ϵ with impurities and phonons, respectively; s is the speed of longitudinal sound. In the derivation of Eq. (2), it was assumed that the energy of phonons interacting with the electrons is small in comparison with the lattice temperature, while their distribution function is equilibrium. A calculation was carried out for an isotropic quadratic spectrum of conduction electrons.

As follows from considerations of symmetry, [6] in the ionic semiconductors InAs, GaAs, InP and so on, which have the structure of zinc blende, the acoustic vibrations can lead to the appearance of a macroscopic electric field. In the scattering of conduction electrons by such a piezoelectric potential, the square of the modulus of the matrix element of interaction has the form [7]

$$|C_{q,\alpha}|^2 = 32\pi^2 \frac{\hbar e^2 \beta^2}{\kappa^2 \rho \omega_{q,\alpha}} [q_1 q_2 e_3(q, \alpha) + q_1 q_3 e_2(q, \alpha) + q_2 q_3 e_1(q, \alpha)]^2 q^{-4}, \quad (3)$$

where β is the piezoelectric modulus of the cubic crystal, ρ is the density, κ is the dielectric constant, $\mathbf{e}(q, \alpha)$ and $\omega_{q,\alpha}$ is the unit polarization vector and the frequency of the phonon with wave vector $\mathbf{q}$ and wave number α , while the indices 1, 2, and 3 indicate the projections of the vectors $\mathbf{q}$ and $\mathbf{e}$ on the principal crystallographic axes.

By means of the method used in [5] for the derivation of (2), and under the same assumptions, it can be shown that the equation for the symmetric part of the distribution $S_0(\epsilon)$ differs from (2) by the substitution

$$\frac{\nu_i(\epsilon)}{\nu_{ph}(\epsilon)} \rightarrow \frac{T}{\epsilon} \frac{\tau_i}{\tau_i}, \quad (4)$$

where τ_i and τ_j are, in order of magnitude, equal to the relaxation times of electrons in their scattering by charged impurities and long-wave piezoelectric phonons in the absence of an electric field. Here s^{-2} has the meaning of the inverse square of the sound velocity, averaged over the directions of its propagation and summed over the polarizations.

Carrying out the substitution (4) in (2), and introducing the non-dimensional energy $z = \epsilon/T$, we represent the function F_0 in the case of interest to us in the form

$$F_0(z) = A \exp \left\{ - \int_0^z dz' \left[1 + \eta \left(1 + \frac{1}{z'} \frac{\tau_i}{\tau_i} \right) \right]^{-1} \right\}, \quad \eta = (cE/sH)^2/3, \quad (5)$$

where A is a normalizing constant.

We have noted above that in almost-elastic scattering of electrons the part of the distribution function f that is even in the velocity of the electron is expressed in terms of F_0 in the same way as in the theory that is linear in E . Therefore, for a fixed concentration of conduction electrons $n = \text{const}$, the current density $\mathbf{j}$ is determined by the expression ($\mathbf{E} \parallel \mathbf{x}$; $\mathbf{H} \parallel \mathbf{z}$)

$$j_x = j_x^{(0)} \int_0^\infty dz \left(\frac{1}{\tau_i} + \frac{z}{\tau_i} \right) \frac{\partial F_0}{\partial z} \Big/ \int_0^\infty dz \left(\frac{1}{\tau_i} + \frac{z}{\tau_i} \right) \frac{\partial f_0}{\partial z}, \quad (6)$$

$$j_y = j_y^{(0)} = nec E/H, \quad (7)$$

where $f_0(z)$ and $\mathbf{j}^{(0)}$ are the electron distribution function and the current density vector, respectively, as $E \rightarrow 0$.

2. We first consider the case $\tau_i \gg \tau_f$, in which the impurities do not play a role in electron scattering. Then

$$F_0(z) = A_0 (1 + \eta)^{-3/2} \exp \{ -z/(1 + \eta) \},$$

$$A_0 = n (2\pi \hbar^2 / mT)^{3/2}, \quad (8)$$

i.e., the electron distribution is described by an exponential with effective temperature

$$T_{eff} = (1 + \eta) T. \quad (9)$$

Substituting (8) in (6) for $\tau_i \gg \tau_f$, it is easy to prove that in this case

$$j_x = j_x^{(0)} \left[1 + \frac{1}{3} (cE/sH)^2 \right]^{-1/2}. \quad (10)$$

Thus, in the region of strong electric fields, $E \gg sH/c$, the density of conduction current j_x is inversely proportional to the third power of the magnetic field H , and does not depend on the value of the electric field E . The saturation of j_x as a function of E is a consequence of the fact that the probability of scattering by a piezoelectric potential is inversely proportional to the square root of the energy of the electron, and, in accord with (8), the mean energy of the electrons for $\eta \gg 1$ is proportional to E^2 .

3. We now consider the case $\tau_i \ll \tau_f$. In the region of electric fields E satisfying the condition

$$\eta\tau_i/\tau_f \ll 1 \ll \eta\tau_f/\tau_i, \quad (11)$$

the scattering of electrons is essentially determined by the charged impurities. In this case, the electron distribution function is

$$F_0(z) \approx A_0 (\tau_i/2\eta\tau_f)^{3/4} \exp(-z^2\tau_i/2\eta\tau_f) \quad (12)$$

and the conduction current density is

$$j_x \approx j_x^{(0)} (\tau_i/2\eta\tau_f)^{3/4} \sim (EH)^{-1/4}. \quad (13)$$

The decrease of the conduction current with increase in the electric field (negative differential conductivity) is explained by the fact that the mean electron energy is

$$\bar{z} \sim \int_0^\infty F_0(z) z^{3/2} dz \sim (\eta\tau_f/\tau_i)^{1/2} \sim E, \quad (14)$$

while the frequency of collisions with charged impurities (and, consequently, the conductivity in a magnetic field) is proportional to $(\bar{z})^{-3/2}$.

Let us make precise the region of applicability of Eq. (13) for j_x . Along with the condition (11), this region is determined by the requirement that the mean collision frequency of electrons with impurities be small in comparison with the cyclotron frequency Ω , i.e.,

$$\Omega\tau_i (\eta\tau_f/\tau_i)^{3/4} \gg 1. \quad (15)$$

In other words, (13) holds also for $\Omega\tau_i \ll 1$, if conditions (11) and (15) are satisfied.

In the experiment one usually measures the resistivity tensor

$$\rho_{ik} = E_i j_k / j^2. \quad (16)$$

In the presence of a strong magnetic field H , the angle between the vectors $\mathbf{E}$ and $\mathbf{j}$ is close to $\pi/2$ ($j_x \ll j_y$). Because of this,

$$j \approx j_y = nec E/H$$

and the dependence of the resistance of the crystal on the value of j has in the case (11) and (15) the form

$$\rho_{xx} \sim j^{-3/2}, \quad \rho_{xy} = H/nec. \quad (17)$$

In Fig. 1, the dependence of the projection of the vector $\mathbf{E}$ in the direction $\mathbf{j}$ is shown as a function of the quantity j . The curve a corresponds to the case $\tau_i \gg \tau_f$, the curve b to the case $\tau_i \ll \tau_f$. Here the same value of τ_f and different values of τ_i correspond to the two curves. The merging of curve b with curve a is a consequence of the fact that, for high values of the current density j , a strong heating of the electron gas takes place and the ionized impurities cease to play a role in the scattering of the electrons.

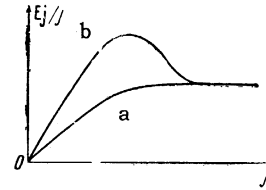


FIG. 1

4. Let us not investigate the electrical conductivity of an ionic semiconductor in the case in which the relaxation of the energy of the electrons is brought about by disperse optical phonons (1) and the relaxation of the momentum by the ionized impurities.

In accord with Davydov and Shmushkevich,^[3] the function $F_0(z)$ for scattering of electrons by optical phonons in the case $z \ll \hbar\omega_0/T$ is determined by the equation

$$eEv(z) = \frac{4}{3} \lambda^2 v_0 \hbar\omega_0 z [F_0(z) + \partial F_0/\partial z], \quad (18)$$

where

$$v(z) = -\frac{2}{3} \frac{eE}{m v_0} z \frac{\partial F_0}{\partial z}$$

is the velocity of the electrons averaged over states with energy z ;

$$v_0 = 2\pi (2m\hbar\omega_0)^{1/2} \left(\frac{\gamma Z e^2}{\hbar\omega_0} \right)^2 \frac{\exp(-\hbar\omega_0/T)}{M a_0^3}$$

is the frequency of collisions of electrons with optical phonons (M is the mass of the elementary

cell, a_0 is the lattice constant, γ is a dimensionless coefficient of polarizability of the ions, Z is the charge); $\lambda = 2m\omega_0 a^2/\hbar$; the quantity $1/\lambda^2\nu_0$ characterizes the electron energy relaxation time.

Equation (18) represents the differential energy balance for electrons with energy z . Actually, the quantity $eE\nu(z)$ is the average power of the electric field absorbed by electrons with energy z , while the right side of (18) is the power transferred by such electrons to the lattice.

In the presence of impurities, the total frequency of collisions of the electron with the scatterers is equal to $\nu_0 + \nu_i(z)$. Therefore, in crossed fields $\mathbf{E} \perp \mathbf{H}$ the velocity of the electron averaged over states with energy z has the form

$$v(z) = -\frac{2}{3} \frac{eE}{m} \frac{\nu_0 + \nu_i(z)}{\Omega^2 + [\nu_0 + \nu_i(z)]^2} z \frac{\partial F_0}{\partial z}. \quad (19)$$

We substitute (19) in Eq. (18) and integrate it in the case $z \gg 1$ and

$$\nu_0 \ll \nu_i(z) \ll \Omega. \quad (20)$$

Bearing in mind that

$$\nu_i(z) \approx 1/\tau_i z^{3/2},$$

we arrive at the following expression for the symmetric part of the distribution function:

$$F_0(z) \approx A_0 g^{-3/2} \exp\left(-\frac{2}{5} g z^{5/2}\right), \quad (21)$$

$$g = \frac{2\Delta}{mc^2} \frac{H^2}{E^2}, \quad (22)$$

where the quantity $\Delta = \lambda^2 \nu_0 \tau_i \hbar \omega_0$ characterizes the energy given off by the electron to the lattice in the time between two successive collisions with impurities.

By means of (21), it is not difficult to compute the mean energy of electrons $\bar{z}$ and the conduction current density:

$$\bar{z} = \int_0^\infty F_0(z) z^{3/2} dz \left/ \int_0^\infty f_0(z) z^{1/2} dz \right. \approx g^{-1/2}, \quad (23)$$

$$j_x \approx j_x^{(0)} g^{3/2} \sim H^{-4/2} E^{-1/2}. \quad (24)$$

Thus a negative differential conductivity occurs also in the case under consideration. The region of applicability of (23) and (24) is determined by the requirements that the mean energy of the electron $\bar{z}$ be much greater than unity and that the mean frequency of collisions with impurities be large in comparison with Ω , i.e., that

$$g \ll 1, \nu_0 \ll g^{3/2} \tau_i^{-1} \ll \Omega. \quad (25)$$

If condition (25) is satisfied the dependence of the resistivity ρ_{xx} on the value of j has the form

$$\rho_{xx} = \sigma_0^{-1} (2\Delta/mc^2)^{3/2} (nec/j)^{1/2}, \quad (26)$$

where σ_0 is the conductivity of the crystal for $H = 0$ and $E \rightarrow 0$.

The dependence of the projection of the electric field $\mathbf{E}$ on the direction of $\mathbf{j}$ as a function of the magnitude of $\mathbf{j}$ is shown in Fig. 2.

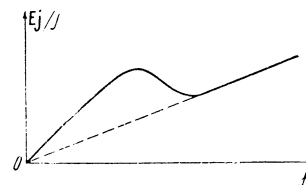


FIG. 2

For small j , Ohm's law is valid and the dependence of $\mathbf{j} \cdot \mathbf{E}/j$ on the value of j is linear. In the case $\Omega\tau_i \gg 1$, the slope of the volt-ampere characteristic is proportional to $1/\tau_i$. For large values of j , heating of the electron gas takes place. As a consequence, the probability of scattering of electrons by ionized impurities decreases and the magnetic resistance of the crystal falls off with increasing j (26). The decreasing portion of the volt-ampere characteristic corresponds to this region of values of j . With further increase in j , the electron gas is heated so much that the probability of scattering of electrons by impurities becomes less than the probability of scattering by optical phonons. Inasmuch as the frequency of collisions with optical phonons ν_0 does not depend on the energy of the electron, the magnetoresistance ceases to depend on the value of j and Ohm's law is again satisfied. However, the slope of the characteristic is now determined by collisions with optical phonons and is proportional to ν_0 and not to $1/\tau_i$.

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SCATTERING OF NEUTRONS BY CRYSTALS WITH IMPURITY NUCLEI AND THE PROBLEM OF RECONSTRUCTING THE VIBRATION SPECTRUM

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The scattering of neutrons by crystals of arbitrary symmetry containing isolated impurities is treated. The cross sections for coherent and incoherent scattering corresponding to one-quantum excitations of the system are found. An analysis is made of the problem of separating out the part of the incoherent cross section which is related to the scattering by the impurity atoms and by the disturbed regions near them. It is shown that this part of the cross section contains direct information about the nature of the vibration of the impurity atom, and, most important, information about the density distribution of the phonon frequency spectrum of the ideal (host) lattice.

1. INTRODUCTION

THE present paper investigates the scattering of neutrons by crystals containing arbitrary impurity nuclei. The concentration of foreign atoms in the host is assumed to be sufficiently small so that they can be considered to be isolated.

If inelastic scattering is regarded as a method for obtaining information about the excitation spectrum in a condensed medium, it makes sense to seek a solution of the inverse problem only. But it is obvious that this requires us to find a consistent solution of the direct problem which gives an explicit relation between the scattering cross section and the spectral characteristics of the medium. This in turn allows an independent solution of the dynamical problem for the macroscopic system.

The problem of the vibrations of a crystal containing an isolated impurity was investigated in its general form in a widely known series of papers by I. Lifshitz (cf., for example, ^[1]) and, for simple models, in papers by Montroll and coworkers. ^[2] This problem has also been studied in detail in connection with studies of the Mössbauer effect. ^[3,4] But for determining the probability of the Mössbauer effect it is necessary only to find the class of solutions to which there corresponds a nonzero vibration amplitude of the impurity atom itself. Assuming that the introduction of the impurity does not change the force constants, the problem is considerably simplified. But in the case of scattering of neutrons, the role of the impurity atom is not limited to the peculiarities of the scattering by that atom alone. In fact the neutrons suffer an additional scattering from the whole region of atoms near the impurity site, whose vibrations

are perturbed. This makes it necessary to find a complete solution of the problem of the vibrations of a lattice with isolated impurity atoms. The detailed treatment of this problem is given in the second section of this paper.

Thus, for low concentrations of impurity atoms there is an additional incoherent scattering from the randomly distributed perturbed regions. This scattering gives information about both the characteristics of the impurity atom and the vibration spectrum of the initial ideal lattice. In the third section of the paper, we find the relation between the cross section for inelastic scattering of neutrons and the two types of information. We also find the conditions for separating the part of the scattering which is due to the presence of impurity atoms.

We treat monatomic lattices of arbitrary symmetry and assume that the impurity atoms are at lattice sites. The problem is solved for an arbitrary ratio of masses of impurity and host atoms on the assumption that the introduction of the impurity does not change the force constants.

Many of the points related to the scattering of neutrons by nonideal crystals have been treated earlier by Krivoglaz. ^[5] He studied the question of the broadening of coherent scattering peaks and the transformation of singularities in the energy spectrum of incoherent neutron scattering. These problems will not be considered here.

2. VIBRATION OF A LATTICE CONTAINING AN IMPURITY ATOM.

A. Consider a lattice with one atom per unit cell. Suppose that there is an impurity atom at

one of the lattice sites (the origin). We shall assume that inserting the impurity does not change the force constants coupling the other atoms to one another and to the foreign atom. Then the equation for determining the eigenfrequencies and eigenfunctions (reduced displacements) for the vibrations of the lattice has the form

$$\omega^2 \Phi^i(\mathbf{r}_n) = \sum_{n'} (m_n m_{n'})^{-1/2} A_0^{ik} (\mathbf{r}_n - \mathbf{r}_{n'}) \Phi^k(\mathbf{r}_{n'}),$$

$$m_n = m(1 - \varepsilon \delta_{n0}) \quad (2.1)$$

with the usual periodicity conditions at the boundary. (The notation is the same as in [3,4]; we sum over repeated Latin superscripts.)

Setting $\varepsilon = 0$ in (2.1), we get the equation for the vibrations of the ideal lattice. The complete system of its $3N$ orthonormal real solutions can be written in the form

$$\varphi_{i\alpha 1}^i(\mathbf{r}_n) = \sqrt{2/N} e^i(\mathbf{f}, \alpha) \cos \mathbf{f} \mathbf{r}_n; \quad (2.2')$$

$$\varphi_{i\alpha-1}^i(\mathbf{r}_n) = \sqrt{2/N} e^i(\mathbf{f}, \alpha) \sin \mathbf{f} \mathbf{r}_n; \quad (2.2'')$$

$$e^i(\mathbf{f}, \alpha) e^i(\mathbf{f}, \alpha') = \delta_{\alpha\alpha'}, \quad \sum_{\alpha=1}^3 e^i(\mathbf{f}, \alpha) e^{i'}(\mathbf{f}, \alpha) = \delta^{ii'},$$

$$\mathbf{e}(-\mathbf{f}, \alpha) = \mathbf{e}(\mathbf{f}, \alpha), \quad \alpha = 1, 2, 3. \quad (2.3)$$

The wave vector $\mathbf{f}$ takes on $N/2$ values uniformly distributed over the region corresponding to half of the fundamental cell in the reciprocal lattice; the index $\zeta = \pm 1$ gives the parity of the solution.

From (2.1) it follows that for $\varepsilon = 0$ the eigenvalues ω_0^2 are independent of ζ , so that an equal number of solutions (2.2') and (2.2'') belong to each frequency.

Let $\{\mathbf{f}\}$ be a set of wave vectors which are connected with one another by the transformations of the point symmetry group of the crystal, the "star" of the irreducible representation. To each star of vectors $\mathbf{f}$ there correspond in general three stars of vectors $\mathbf{e}(\mathbf{f}, \alpha)$ ($\alpha = 1, 2, 3$) and three different eigenvalues. We shall number the vectors of a star by the index λ , and the different stars of vectors $\mathbf{e}$ by the index p . Then instead of the usual dependence of the squared frequency and polarization vector on wave vector and branch number

$$\omega_0^2(\mathbf{f}, \alpha), \quad \mathbf{e}(\mathbf{f}, \alpha)$$

we have

$$\omega_{0p\lambda}^2, \quad \mathbf{e}_{p\lambda},$$

where eigenvalues with the same λ coincide:

$$\omega_{0p\lambda}^2 = \omega_{0p}^2, \quad \lambda = 1, 2, \dots, \vartheta_p/2.$$

We shall assume that the frequencies are labelled by the index p in increasing order. Twice the

number of vectors $\mathbf{e}$ in a star obviously gives the degree of degeneracy ϑ_p of the corresponding frequency (the factor two is due to the degeneracy in ζ).

For an overwhelmingly large number of the frequencies this multiplicity has the same value ϑ , given by the symmetry of the lattice. The exceptions occur when the wave vector $\mathbf{f}$ to which the frequency corresponds is distinguished by some symmetry element of the crystal (for example, is in a symmetry plane or along a symmetry axis), or when there is accidental degeneracy. But we shall be interested only in quantities which are obtained by summation over a macroscopic number of states. We shall therefore not use the index λ in the stars of $\mathbf{f}$ and $\mathbf{e}(\mathbf{f}, \alpha)$.

In the presence of the impurity there is a partial lifting of the degeneracy of all the levels. The frequencies which are split off fall in the interval between ω_{0p} and ω_{0p+1} . As was shown in [3], Eq. (2.1) can be transformed, for the frequencies which are split, to the following form:

$$D^{ik}(\omega^2) j^k = j^i, \quad (2.4)$$

where $\mathbf{j}$ is a unit vector in the direction of the displacement of the impurity atom ($\Phi(0) = \Phi \mathbf{j}$), while

$$D^{ik}(\omega^2) = \frac{2\varepsilon\omega^2}{N} \sum_{\mathbf{f}\alpha} \frac{e^i(\mathbf{f}, \alpha) e^k(\mathbf{f}, \alpha)}{\omega^2 - \omega_0^2(\mathbf{f}, \alpha)}. \quad (2.5)$$

Throughout the paper, summations and integrations extend over half the fundamental cell of the reciprocal lattice (this is the reason for the factor of 2 compared to [1]).

An analysis of (2.4) shows that the maximum number of frequencies split off from each level is three, the actual number depending on the symmetry of the crystal. Let us consider the various crystal systems.

1. Triclinic system. There is one non-identical symmetry transformation, the inversion. All frequencies ω_0 are degenerate with the same multiplicity $\vartheta = 2$. In the presence of the impurity atom, one frequency is split off from each degenerate level, corresponding to the value $\mathbf{j}(\omega^2)$ determined from (2.4).

2. Monoclinic system. The degree of degeneracy is $\vartheta = 4$. The symmetric tensor D^{ik} has four independent components. In the general case, two frequencies are split off—one belongs to a vibration with $\mathbf{j}$ parallel to the two-fold axis, the other is determined at the same time has its $\mathbf{j}$ in the plane perpendicular to the two-fold axis.

3. Rhombic system. The degree of degeneracy is $\vartheta = 8$. In the coordinate system with its axes along the two-fold symmetry axes, the tensor D^{ik}

is in diagonal form. Then (2.4) splits into three independent equations. We thus get three frequencies splitting off from each level, with polarization vectors $\mathbf{j}$ along the symmetry axes.

4. Tetragonal, trigonal, and hexagonal systems.

In all these cases, the tensor $\mathbf{D}_{ik}$ reduces to diagonal form in a coordinate system having one of its axes along the c axis. Two of the three equations which correspond to the other two axes coincide. Thus there is a splitting off of one frequency with $\mathbf{j}$ parallel to the c axis, and two identical frequencies with mutually perpendicular vectors $\mathbf{j}$ lying in an arbitrary position in the plane perpendicular to the c axis.

5. Cubic system. From each degenerate level, three identical frequencies split off; the system of three mutually perpendicular vectors $\mathbf{j}$ can be chosen arbitrarily.

In accordance with the above considerations, we shall not deal in detail with cases where the directions of the vectors of the star are distinguished crystallographically, although the corresponding analysis can be done in straight-forward fashion. We mention only that for an initial frequency for which the star of vectors $\mathbf{e}$ lies in a symmetry plane, after introduction of the impurity atom two frequencies are split off; for frequencies to which there corresponds a star of vectors $\mathbf{e}$ directed along a symmetry axis, only one frequency splits off, independent of the symmetry of the crystal.

To simplify the presentation, we shall restrict ourselves in what follows to crystals with symmetry no lower than rhombic. In all these cases the number of frequencies split off is three.

B. As we have already remarked, the impurity atom participates only in vibrations whose frequencies are split off. Its displacement in all other vibrations with unchanged frequencies ω_0 is zero. As a result, for all the unsplit frequencies one can easily show that (2.1) is identical with the equation for the vibrations of the ideal lattice.

The complete system of orthonormal functions for the ideal lattice has the form (2.2). We replace $\mathbf{f}$, α , ξ by the indices p , ν , where

$$\nu = \begin{cases} \lambda & \xi = 1 \\ \lambda + \vartheta/2 & \xi = -1. \end{cases}$$

Obviously,

$$\mathbf{f}_{\lambda+\vartheta/2} = \mathbf{f}_\lambda$$

and consequently

$$\mathbf{e}_{p\lambda+\vartheta/2} = \mathbf{e}_{p\lambda}, \quad \omega_{0p\lambda+\vartheta/2}^2 = \omega_{0p\lambda}^2 = \omega_{0p}^2.$$

Then (2.2) can be rewritten as

$$\varphi_{p\nu}^i(\mathbf{r}_n) = \sqrt{2/N} e_{p\nu}^i \cos \mathbf{f}_{p\nu} \mathbf{r}_n, \quad \nu = 1, 2, \dots, \vartheta/2; \quad (2.6')$$

$$\varphi_{p\nu}^i(\mathbf{r}_n) = \sqrt{2/N} e_{p\nu}^i \sin \mathbf{f}_{p\nu} \mathbf{r}_n, \quad \nu = \vartheta/2 + 1, \dots, \vartheta. \quad (2.6'')$$

The total number of independent functions in (2.2) and (2.6) is, of course, the same and equal to $3N$.

We make an orthogonal transformation and replace (2.6) by a new system of functions:

$$\begin{aligned} \psi_{p\nu}^i(\mathbf{r}_n) &= \sum_{\delta=1}^{\vartheta/2} S_{\nu\delta}(p) \varphi_{p\delta}^i(\mathbf{r}_n), \quad \nu = 1, 2, \dots, \vartheta/2; \\ \varphi_{p\nu}^i(\mathbf{r}_n), \quad \nu &= \vartheta/2 + 1, \dots, \vartheta \end{aligned} \quad (2.7)$$

and require that for $\nu = 4, \dots, \vartheta/2$

$$\psi_{p\nu}^i(0) = 0$$

or

$$\sum_{\delta=1}^{\vartheta/2} S_{\nu\delta}(p) e_{p\delta}^i = 0. \quad (2.8)$$

It is easy to show that an orthogonal matrix satisfying the $3(\vartheta/2 - 3)$ ($i = 1, 2, 3$) conditions (2.8) can be constructed for any crystal with symmetry equal to rhombic or higher; this matrix is determined to within an orthogonal transformation which leaves invariant the subspaces of functions (2.7) with $\nu = 1, 2, 3$ [cf. (2.15)] and with $\nu = 4, \dots, \vartheta/2$ [and consequently also the conditions (2.8)]. For crystals of the triclinic and monoclinic systems ($\vartheta/2 < 3$), all the possible solutions of equation (2.1) are exhausted by the functions (2.6'') and the functions corresponding to the split-off frequencies [cf. later in (2.17)], so that there is no need for the construction given here.

Thus for each p we have determined $\vartheta - 3$ functions $\psi_{p\nu}(\mathbf{r}_n)$ ($\nu = 4, \dots, \vartheta/2$) and $\varphi_{p\nu}(\mathbf{r}_n)$ ($\nu = \vartheta/2 + 1, \dots, \vartheta$), which are eigenfunctions of the vibration problem for the ideal lattice, correspond to ω_{0p}^2 , and vanish for $\mathbf{r}_n = 0$.

We now find the explicit form of the functions $\psi_{p\nu'}^i(\mathbf{r}_n)$ for $\nu' = 1, 2, 3$. The condition that they be orthogonal to the functions (2.7) with $\nu = 4, \dots, \vartheta/2$ gives the relation

$$\sum_n \psi_{p\nu'}^i(\mathbf{r}_n) \psi_{p\nu}^i(\mathbf{r}_n) = \sum_{\delta=1}^{\vartheta/2} S_{\nu\delta}(p) S_{\nu\delta}(p) = 0. \quad (2.9)$$

We choose $S_{\nu'\delta}(p)$ in the form

$$S_{\nu'\delta}(p) = C_{\nu'}(p) e_{p\delta}^i j_{p\nu'}^i, \quad (2.10)$$

where $\mathbf{j}_{p\nu'}$ is an arbitrary unit vector. Then, if we use (2.8) we see that condition (2.9) is satisfied identically.

Let $\nu'' = 1, 2, 3$. Then

$$\sum_n \psi_{p\nu'}^i(\mathbf{r}_n) \psi_{p\nu''}^i(\mathbf{r}_n) = C_{\nu'}(p) C_{\nu''}(p) \sum_{\delta=1}^{\vartheta/2} e_{p\delta}^i e_{p\delta}^k j_{p\nu'}^i j_{p\nu''}^k. \quad (2.11)$$

We introduce the tensor

$$L^{ik}(p) = \sum_{\delta=1}^{\vartheta/2} e_{p\delta}^i e_{p\delta}^k. \quad (2.12)$$

We choose the vectors $\mathbf{j}_{p\nu'}$ so that they are eigenvectors of the tensor $L^{ik}(p)$, i.e.,

$$L^{ik}(p) j_{p\nu'}^k = \eta_{\nu'}(p) j_{p\nu'}^i, \quad \nu' = 1, 2, 3. \quad (2.13)$$

Because they are mutually orthogonal, (2.11) vanishes for $\nu' \neq \nu''$, while the condition for normalization of the functions (2.7) to unity gives the relation

$$C_{\nu'}(p) = 1/\sqrt{\eta_{\nu'}(p)}. \quad (2.14)$$

Thus

$$\psi_{p\nu}^i(\mathbf{r}_n) = [\eta_{\nu}(p)]^{-1/2} \sum_{\delta=1}^{\vartheta/2} (\mathbf{j}_{\nu} \mathbf{e}_{p\delta}) \Phi_{p\delta}^i(\mathbf{r}_n), \quad \nu = 1, 2, 3. \quad (2.15)$$

Starting with the rhombic system, the vectors $\mathbf{j}_{p\nu} = \mathbf{j}_{\nu}$ are along the fixed principal axes of the crystal. Then

$$\sum_{\nu=1}^3 \eta_{\nu}(p) = L^{ii}(p) = \vartheta/2.$$

In particular, in a cubic crystal

$$L^{ik}(p) = 1/3 \delta^{ik}, \quad \eta_{\nu}(p) = \vartheta/6. \quad (2.16)$$

The functions (2.7) and (2.6'') form a complete system of eigenfunctions for the problem of vibration of an ideal lattice. In the presence of the impurity atom, the system of eigenfunctions changes. It is now formed by the functions (2.6'') and the functions (2.7) for $\nu = 4, \dots, \vartheta/2$ (taking (2.8) into account), while new functions corresponding to the split-off frequencies appear in place of (2.15). These functions, which are actually superpositions of the functions (2.15) with different p and ν , were found in [3] and can be written, taking account of normalization, in the following form:

$$\Phi_{p\nu}^i(\mathbf{r}_n) = \left(\frac{d \ln \omega_{p\nu}^2}{d\varepsilon} \right)^{1/2} j_{\nu}^k \frac{2}{N} \sum_{\mathbf{f}\alpha} \left[\varepsilon \omega_{p\nu}^2 \frac{e^i(\mathbf{f}, \alpha) e^k(\mathbf{f}, \alpha)}{\omega_{p\nu}^2 - \omega_0^2(\mathbf{f}, \alpha)} + \frac{1}{3} (\sqrt{1-\varepsilon} - 1) \delta^{ik} \right] \cos \mathbf{f} \mathbf{r}_n, \quad \nu = 1, 2, 3; \quad (2.17)$$

$$\frac{d \ln \omega_{p\nu}^2}{d\varepsilon} = \left\{ \frac{2\varepsilon^2 \omega_{p\nu}^4}{N} \sum_{\mathbf{f}\alpha} \frac{(\mathbf{j}_{\nu} \mathbf{e}(\mathbf{f}, \alpha))^2}{[\omega_{p\nu}^2 - \omega_0^2(\mathbf{f}, \alpha)]^2} - \varepsilon \right\}^{-1}. \quad (2.18)$$

Thus if we use $\Phi_{p\nu}^i(\mathbf{r}_n)$ to denote the complete system of orthonormal functions for the problem of the vibration of the lattice in the presence of the impurity atom, then for $\nu = 1, 2, 3$, these functions coincide with (2.17) (the corresponding eigenvalues are $\omega_{p\nu}^2$), for $\nu = 4, \dots, \vartheta/2$ they coincide with (2.7)–(2.8), and for $\nu = \vartheta/2 + 1, \dots, \vartheta$, with (2.6'') ($\omega_{p\nu}^2 = \omega_{0p}^2$ for $4 \leq \nu \leq \vartheta$).

C. We shall again limit ourselves to crystals with symmetry no lower than rhombic. Then, from (2.4) and (2.5), in a coordinate system whose axes are along the principal symmetry axes of the crystal,

we have three independent equations for determining the shifted frequencies:

$$\frac{2\varepsilon \omega_{p\nu}^2}{N} \sum_{\mathbf{f}\alpha} \frac{[e^{\nu}(\mathbf{f}, \alpha)]^2}{\omega_{p\nu}^2 - \omega_0^2(\mathbf{f}, \alpha)} = 1, \quad \nu = 1, 2, 3. \quad (2.19)$$

These equations, in addition to the quasicontinuous spectrum $0 < \omega_{p\nu}^2 < \omega_{0\max}^2$ ($\omega_{0\max}$ is the limiting frequency of the quasicontinuous spectrum of the regular lattice), can have solutions in the form of discrete frequencies $\omega_{p\nu}^2 > \omega_{0\max}^2$.

For $\omega^2 > \omega_{0\max}^2$ the denominator in (2.19) does not vanish, and we can change from summation over $\mathbf{f}$ to integration. As a result, to determine the discrete frequencies, if they exist, we have an equation analogous to the one found by I. Lifshitz [1]:

$$\varepsilon \omega^2 \int_0^{\omega_{0\max}^2} \frac{g^{(\nu)}(\omega_0^2) d\omega_0^2}{\omega^2 - \omega_0^2} = 1, \quad \nu = 1, 2, 3, \quad (2.20)$$

where

$$g^{(\nu)}(\omega_0^2) = \frac{2v_0}{(2\pi)^3} \sum_{\alpha} \int_{\omega_0^2(\mathbf{f}, \alpha) = \omega_0^2} \frac{[e^{\nu}(\mathbf{f}, \alpha)]^2}{|\nabla \omega_0^2(\mathbf{f}, \alpha)|} d\Omega_{\mathbf{f}} \quad (2.21)$$

(v_0 is the volume of the unit cell). The integration in (2.21) extends over those portions of the surface $\omega_0^2(\mathbf{f}, \alpha) = \omega_0^2$ which are located within the region chosen initially—half the fundamental cell of the reciprocal lattice.

The quantity

$$g(\omega_0^2) = \frac{1}{3} \sum_{\nu=1}^3 g^{(\nu)}(\omega_0^2) = \frac{2v_0}{3(2\pi)^3} \sum_{\alpha} \int_{\omega_0^2(\mathbf{f}, \alpha) = \omega_0^2} \frac{d\Omega_{\mathbf{f}}}{|\nabla \omega_0^2(\mathbf{f}, \alpha)|} \quad (2.22)$$

is a function of the density of the squared frequencies for the initial ideal lattice. From the very definitions (2.21) and (2.22) it follows that the functions $g^{(\nu)}(\omega_0^2)$ and $g(\omega_0^2)$ satisfy the normalization

$$\int_0^{\omega_{0\max}^2} g^{(\nu)}(\omega_0^2) d\omega_0^2 = 1, \quad \int_0^{\omega_{0\max}^2} g(\omega_0^2) d\omega_0^2 = 1. \quad (2.23)$$

All the considerations about the degeneracy of the split-off frequencies obviously remain valid for the discrete frequencies. Thus in a cubic crystal $g^{(\nu)}(\omega_0^2) = g(\omega_0^2)$, and we arrive at the single equation (2.20) for determining the three-fold degenerate frequency.

It is easy to show that each of the equations (2.20) has a unique solution if and only if

$$\varepsilon > \left[\int_0^{\omega_{0\max}^2} \frac{g^{(\nu)}(\omega_0^2) d\omega_0^2}{1 - \omega_0^2/\omega_{0\max}^2} \right]^{-1}. \quad (2.24)$$

This inequality is the condition which the spectrum of the system must satisfy in order to have discrete frequencies. Their number can vary up to three,

depending on the values of ν for which (2.24) is valid.

Because the right side of (2.24) is positive, discrete frequencies appear only when the impurity atoms are lighter than the host atoms, and for the usual vibration spectra m_0/m must be considerably less than unity. (For example, in a simple cubic lattice, with nearest neighbor interactions, and with equal central and noncentral force constants, the discrete frequency appears only when $m_0/m < 0.670$.)

D. In the quasicontinuous spectrum, as in [1,3], we represent $\omega_{p\nu}^2$ in the form

$$\begin{aligned} \omega_{p\nu}^2 &= \omega_{0p}^2 + \Delta\omega_{p\nu}^2, & \Delta\omega_{p\nu}^2 &= z_{p\nu} \delta\omega_{0p}^2, \\ \delta\omega_{0p}^2 &= \omega_{0p+1}^2 - \omega_{0p}^2, & 0 &\leq z_{p\nu} \leq 1. \end{aligned} \quad (2.25)$$

Taking a quasiequidistant distribution of the squared frequencies ω_{0p}^2 , we find the value of $z_{p\nu}$ from Eq. (2.19) by the usual method (cf. [1,3]). Going to the limit $\delta\omega_{0p}^2 \rightarrow 0$, $N \rightarrow \infty$, so that $\omega_{0p}^2 = \omega^2$ is unshifted, we finally get

$$z_\nu(\omega^2) = \frac{1}{\pi} \arctg \left\{ \left[1 - \varepsilon \omega^2 \int_0^{\omega_0^2 \max} \frac{g^{(\nu)}(\omega_0^2) d\omega_0^2}{\omega^2 - \omega_0^2} \right] \frac{1}{\pi \varepsilon \omega^2 g^{(\nu)}(\omega^2)} \right\}. \quad (2.26)^*$$

This result can be used immediately for computing quantities having the form of sums over the shifted frequencies of the quasicontinuous spectrum with weights $d \ln \omega_{p\nu}^2 / d\varepsilon$. Using the relation

$$\frac{d \ln \omega_{p\nu}^2}{d\varepsilon} = \frac{\delta\omega_{0p}^2}{\omega_{0p}^2} \frac{\partial z_{p\nu}}{\partial \varepsilon},$$

which follows from (2.25), we quickly get

$$\lim_{N \rightarrow \infty} \sum_{\substack{p \\ (\omega^2=0)}}^{(\omega_0^2 \max)} \frac{d \ln \omega_{p\nu}^2}{d\varepsilon} (\dots) = \int_0^{\omega_0^2 \max} d\omega^2 f_\nu(\omega^2) (\dots), \quad (2.27)$$

where

$$\begin{aligned} f_\nu(\omega^2) &\equiv \frac{1}{\omega^2} \frac{\partial z_\nu(\omega^2)}{\partial \varepsilon} \\ &= g^{(\nu)}(\omega^2) \left\{ \left[1 - \varepsilon \omega^2 \int_0^{\omega_0^2 \max} \frac{g^{(\nu)}(\omega_0^2) d\omega_0^2}{\omega^2 - \omega_0^2} \right]^2 + [\pi \varepsilon \omega^2 g^{(\nu)}(\omega^2)]^2 \right\}^{-1} \end{aligned} \quad (2.28)$$

(assuming, of course, that the integral in (2.27) exists).

One can get an important normalization relation for the quantity $f_\nu(\omega^2)$. To derive it, we set $\mathbf{r}_n = \mathbf{r}_{n'}$, $i = i'$ (no summation!) in the relation

$$\sum_{p\nu} \Phi_{p\nu}^i(\mathbf{r}_n) \Phi_{p\nu}^{i'}(\mathbf{r}_{n'}) = \delta^{ii'} \delta_{\mathbf{r}_n \mathbf{r}_{n'}}, \quad (2.29)$$

which follows from the completeness of the ortho-

normal function system $\Phi_{p\nu}^i(\mathbf{r}_n)$. Remembering that

$$\Phi_{p\nu}(0) = \begin{cases} \sqrt{1 - \varepsilon} (d \ln \omega_{p\nu}^2 / d\varepsilon)^{1/2} j_\nu, & \nu = 1, 2, 3 \\ 0 & \nu = 4, \dots, \infty \end{cases} \quad (2.30)$$

[cf. (2.17) and (2.4)], and also that $j_\nu^i = \delta^{i\nu}$, we get

$$(1 - \varepsilon) \sum_p d \ln \omega_{p\nu}^2 / d\varepsilon = 1. \quad (2.31)$$

Changing from summation to integration over the quasicontinuous spectrum, we get the required relation

$$(1 - \varepsilon) \int_0^{\omega_0^2 \max} f_\nu(\omega^2) d\omega^2 + (1 - \varepsilon) d \ln \omega_{D\nu}^2 / d\varepsilon = 1. \quad (2.32)$$

The second term comes from the discrete frequency. As expected, for $\varepsilon = 0$ (2.32) goes over into the first of the conditions (2.23).

If we know the partial density function for the squared frequencies $g^{(\nu)}(\omega_0^2)$, the quantity $d \ln \omega_{D\nu}^2 / d\varepsilon$ as a function of $\omega_{D\nu}^2$ is given by

$$\frac{d \ln \omega_{D\nu}^2}{d\varepsilon} = \left\{ \varepsilon^2 \omega_{D\nu}^2 \int_0^{\omega_0^2 \max} \frac{g^{(\nu)}(\omega_0^2) d\omega_0^2}{(\omega_{D\nu}^2 - \omega_0^2)^2} - \varepsilon \right\}^{-1}, \quad (2.33)$$

which follows immediately from (2.18) when we change to integration.

The numerical value of $d \ln \omega_{D\nu}^2 / d\varepsilon$ as a function of ε alone can be found in principle from (2.32) as an integral of $f_\nu(\omega^2)$, or by determining $\omega_{D\nu}^2 = \omega_{D\nu}^2(\varepsilon)$ from Eq. (2.20).

E. Let us write the displacement of an arbitrary atom as an expansion in normal vibrations:

$$u^i(\mathbf{r}_n) = \sum_{p\nu} u_{p\nu}^i(\mathbf{r}_n). \quad (2.34)$$

It is not difficult to show that

$$u_{p\nu}^i(\mathbf{r}_n) = \sqrt{\hbar / m_n \omega_{p\nu}} \Phi_{p\nu}^i(\mathbf{r}_n) q_{p\nu}, \quad (2.35)$$

where $q_{p\nu}$ is a dimensionless real normal coordinate, corresponding to the vibration with frequency $\omega_{p\nu}$.

3. SCATTERING OF SLOW NEUTRONS; DISCRETE FREQUENCIES ABSENT

Let us consider the scattering of neutrons by crystals with impurity atoms. We assume that the concentration of impurities is so low that the scattering occurs independently from the isolated regions of disturbance around the impurity atoms. We can then restrict the treatment to neutron scattering by a crystal with a single impurity nucleus, and make use of the results obtained above.

The general expression for the interaction of a neutron with an arbitrary crystal lattice can be written as follows (cf. [6]):

*tg = tan.

$$V(\mathbf{r}) = \sum_n [A_n \hat{I} + B_n (\hat{S} \hat{K}_n)] \delta(\mathbf{r} - \mathbf{R}_n). \quad (3.1)$$

Here $\hat{S}$ is the neutron spin operator; $\hat{K}_n$ is the spin operator for the nucleus at the n -th lattice site; A_n and B_n are nuclear parameters; $\mathbf{R}_n = \mathbf{r}_n + \mathbf{u}(\mathbf{r}_n)$.

Let us consider the matrix element of (3.1) corresponding to a transfer of momentum $\hbar \mathbf{k} = \hbar(\mathbf{k}_1 - \mathbf{k}_2)$ to the lattice with a simultaneous change in occupation numbers of $r_{p\nu} = n'_{p\nu} - n_{p\nu}$. Remembering (2.35), after averaging over the initial equilibrium state of the crystal and averaging over the initial and summing over the final states for the projection of the neutron and nuclear spins, we find for the square modulus of the matrix element

$$\begin{aligned} M(\{r_{p\nu}\}, \mathbf{k}) &= \sum_n \sum_{n'} A_n A_{n'} \exp[i\mathbf{k}(\mathbf{r}_n - \mathbf{r}_{n'})] \exp\left(-\frac{\omega_n + \omega_{n'}}{2}\right) \\ &\times \prod_{p\nu} \exp\left(\frac{r_{p\nu} \hbar \omega_{p\nu}}{T}\right) I_{r_{p\nu}} \left[\frac{\hbar(\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))(\mathbf{k} \Phi_{p\nu}(\mathbf{r}_{n'}))}{2 \sqrt{m_n m_{n'} \omega_{p\nu}}} \operatorname{sh}^{-1} \frac{\hbar \omega_{p\nu}}{2T} \right] \\ &+ \frac{1}{4} \sum_n B_n^2 K_n (K_n + 1) e^{-\omega_n} \prod_{p\nu} \exp\left(\frac{r_{p\nu} \hbar \omega_{p\nu}}{2T}\right) I_{r_{p\nu}} \\ &\times \left[\frac{\hbar(\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2}{2 m_n \omega_{p\nu}} \operatorname{sh}^{-1} \frac{\hbar \omega_{p\nu}}{2T} \right], \end{aligned} \quad (3.2)^*$$

$$\omega_n = \frac{\hbar}{2m_n} \sum_{p\nu} \frac{(\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2}{\hbar \omega_{p\nu}} (2\bar{n}_{p\nu} + 1). \quad (3.3)$$

We have used the relation^[4]

$$\begin{aligned} \langle n+r | \exp[i\sqrt{2x}q] | n \rangle &= \begin{cases} e^{-x/2} (i\sqrt{x})^r \sqrt{n!/(n+r)!} L_n^r(x), & r > 0 \\ e^{-x/2} (i\sqrt{x})^{|r|} \sqrt{(n+r)!/n!} L_{n+r}^{|r|}(x), & r \leq 0 \end{cases} \end{aligned}$$

and the summation formula^[7]

$$\begin{aligned} (1-z) \sum_{n=0}^{\infty} \frac{n!}{(n+r)!} L_n^r(x) L_n^r(y) z^n &= \exp\left\{-\frac{(x+y)z}{1-z}\right\} (xyz)^{-r/2} I_r\left\{\frac{2(xy)z}{1-z}\right\}. \end{aligned}$$

Here L_n^r is a generalized Laguerre polynomial, I_r is the Bessel function of r -th order and pure imaginary argument, T is the crystal temperature in energy units.

We note that the expressions (3.2)–(3.3) are valid for the scattering of a neutron by an arbitrary system of harmonically interacting particles. The particular properties of the system find their

expression in the explicit form of the functions $\Phi_{p\nu}^i(\mathbf{r}_n)$ and the spectrum of frequencies $\omega_{p\nu}^2$.

In this section we assume that there are no localized vibrations with discrete frequencies, i.e., there is only a quasicontinuous spectrum. Remembering that the amplitudes of the atomic displacements in each normal vibration belonging to the quasicontinuous spectrum $\sim N^{-1/2}$, we should restrict ourselves to cases where all the $|r_{p\nu}| \leq 1$. Then

$$\exp\left\{\pm \frac{\hbar \omega}{2T}\right\} I_{\pm 1}\left[(xy)^{1/2} \frac{1}{\operatorname{sh}(\hbar \omega/2T)}\right] = (xy)^{1/2} \left(\bar{n} + \frac{1}{2} \pm \frac{1}{2}\right). \quad (3.4)$$

In addition suppose that the regular lattice consists of quite heavy nuclei. Then at not too high temperatures we can neglect the contribution of multiquantum transitions and treat only scattering processes which are accompanied by the creation or absorption of one quantum of the vibration p, ν . For such a process, using (3.4), Eq. (3.2) takes the form

$$\begin{aligned} M_{p\nu} &= \left\{ \left| \sum_n \frac{A_n}{\sqrt{m_n}} e^{-\omega_n/2} e^{i\mathbf{k}\mathbf{r}_n} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) \right|^2 \right. \\ &+ \frac{1}{4} \sum_n \frac{B_n^2}{m_n} K_n (K_n + 1) e^{-\omega_n} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2 \left. \right\} \\ &\times \frac{\hbar}{2\omega_{p\nu}} \left(\bar{n}_{p\nu} + \frac{1}{2} \pm \frac{1}{2}\right). \end{aligned} \quad (3.5)$$

(From here on the upper sign refers to creation, the lower to absorption by the neutron of a vibration quantum.)

Let us make some remarks about the quantity $e^{-\omega_n}$ in (3.3), the Debye-Waller factor. At sufficiently high temperatures, when

$$T \gg \hbar \omega_{0 \max} \sim \Theta \quad (3.6)$$

(Θ is the Debye temperature), the relation

$$2\bar{n}_{p\nu} + 1 \approx \frac{2T}{\hbar \omega_{p\nu}} \left[1 + \frac{1}{12} \left(\frac{\hbar \omega_{p\nu}}{T} \right)^2 \right]. \quad (3.7)$$

holds. We then have for w_n ,

$$w_n = 2TR_n \frac{1}{\hbar^2} \sum_{p\nu} \frac{1}{\omega_{p\nu}^2} (\mathbf{x} \Phi_{p\nu}(\mathbf{r}_n))^2 + \frac{R_n}{6T} \sum_{p\nu} (\mathbf{x} \Phi_{p\nu}(\mathbf{r}_n))^2. \quad (3.8)$$

Here R_n is the recoil energy of atom n :

$$R_n = \frac{\hbar^2 k^2}{2m_n} = R \frac{1}{(1 - \varepsilon \delta_{n0})}, \quad R = \frac{\hbar^2 k^2}{2m}, \quad \mathbf{x} = \frac{\mathbf{k}}{k}. \quad (3.9)$$

Because of (2.29), the last term in (3.8) is simply equal to $R_n/6T$. To calculate the first term in (3.8), in complete analogy with the procedure in^[4], we multiply both sides of (2.1) (for the p, ν vibration) by

$$\omega_{p\nu}^{-2} \Phi_{p\nu}^k(\mathbf{r}_i) (m_l m_n)^{1/2} [A_0^{-1}(\mathbf{r}_l - \mathbf{r}_n)]^i$$

*sh = sinh.

$([A_0^{-1}(\mathbf{r}_l - \mathbf{r}_n)]^{ik})$ is the matrix inverse to $A_0^{ik}(\mathbf{r}_l - \mathbf{r}_n)$ and sum over $\mathbf{r}_n$, i , p , ν . Using (2.29), we get

$$\sum_{p\nu} \omega_{p\nu}^{-2} \Phi_{p\nu}^i(\mathbf{r}_l) \Phi_{p\nu}^k(\mathbf{r}_l) = m_l [A_0^{-1}(0)]^{ik}. \quad (3.10)$$

A similar relation holds for the original ideal lattice:

$$\begin{aligned} \sum_{i\alpha\zeta} \omega_0^{-2}(\mathbf{f}, \alpha) \Phi_{i\alpha\zeta}^i(\mathbf{r}_l) \Phi_{i\alpha\zeta}^k(\mathbf{r}_l) \\ = \frac{2}{N} \sum_{i\alpha} \omega_0^{-2}(\mathbf{f}, \alpha) e^i(\mathbf{f}, \alpha) e^k(\mathbf{f}, \alpha) = m [A_0^{-1}(0)]^{ik} \end{aligned} \quad (3.11)$$

Comparing (3.8), (3.10), and (3.11) we finally get

$$\omega_n = 4RT \frac{1}{\hbar^2} \frac{1}{N} \sum_{i\alpha} \omega_0^{-2}(\mathbf{f}, \alpha) (\kappa \mathbf{e}(\mathbf{f}, \alpha))^2 + \frac{R_n}{6T}. \quad (3.12)$$

In the case of a cubic crystal, the quantity w_n becomes isotropic and takes the form

$$w_n = \frac{2RT}{\hbar^2} \langle \omega_0^{-2} \rangle + \frac{R_n}{6T}, \quad \langle \omega_0^{-2} \rangle = \frac{2}{3N} \sum_{i\alpha} \omega_0^{-2}(\mathbf{f}, \alpha). \quad (3.12')$$

Thus, neglecting the second term in (3.12) (which is already small for $T \sim \hbar\omega_0 \max$), we see that the Debye-Waller factor is independent of n when (3.6) is satisfied.

On the other hand, when $T < \hbar\omega_0 \max$, for a lat-

tice consisting of sufficiently heavy atoms, such as we are considering, w_n is itself small, and in this case there is a small error if we neglect its dependence on n . In accordance with this, we assume throughout the constancy of w_n for all n .

We average (3.5) over the isotopic composition of the host lattice, using the standard method (cf., for example, [6]). Denoting quantities referring to the impurity nucleus by the subscript 0, we get

$$\begin{aligned} \overline{M}_{p\nu} = e^{-w} \left\{ \frac{1}{m} \overline{A}^2 \left| \sum_{n \neq 0} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) e^{i\mathbf{k}\mathbf{r}_n} \right|^2 + \frac{1}{m} C \sum_n (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2 \right. \\ \left. + \frac{2}{(mm_0)^{1/2}} A_0 \overline{A} (\mathbf{k} \Phi_{p\nu}(0)) \sum_{n \neq 0} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) \cos \mathbf{k}\mathbf{r}_n \right. \\ \left. + \left[\frac{1}{m_0} (A_0^2 + \frac{1}{4} B_0^2 K_0 (K_0 + 1)) \right. \right. \\ \left. \left. - \frac{1}{m} C \right] (\mathbf{k} \Phi_{p\nu}(0))^2 \right\} \frac{\hbar (\bar{n}_{p\nu} + 1/2 \pm 1/2)}{2\omega_{p\nu}}, \end{aligned} \quad (3.13)$$

$$C = \overline{A}^2 - \overline{A}^2 + 1/4 \overline{B}^2 K (K + 1). \quad (3.14)$$

First let us consider the third and fourth terms in (3.13). They differ from zero only for $\nu = 1, 2, 3$, since only for these values is $\Phi_{p\nu}(0) \neq 0$. Substituting (2.17) and (2.30) in the third and fourth terms in (3.13), after simple transformations we arrive at the expression

$$\frac{1}{m} \frac{d \ln \omega_{p\nu}^2}{d\varepsilon} \left\{ 2\overline{A} A_0 (\mathbf{j}, \mathbf{k}) \left[\varepsilon \omega_{p\nu}^2 \sum_{\alpha} \frac{(\mathbf{k}\mathbf{e}(\mathbf{k}, \alpha)) (\mathbf{j}\mathbf{e}(\mathbf{k}, \alpha))}{\omega_{p\nu}^2 - \omega_0^2(\mathbf{k}, \alpha)} - \mathbf{j}, \mathbf{k} \right] + (\mathbf{j}, \mathbf{k})^2 [A_0^2 + 1/4 B_0^2 K_0 (K_0 + 1) - (1 - \varepsilon) C] \right\}, \quad \nu = 1, 2, 3. \quad (3.15)$$

For the first and second terms in (3.13), using (2.6)–(2.8) and (2.17), we find respectively

$$\left| \sum_{n \neq 0} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) e^{i\mathbf{k}\mathbf{r}_n} \right|^2 = \begin{cases} \frac{d \ln \omega^2}{d\varepsilon} \left[\varepsilon \omega^2 \sum_{\alpha} \frac{(\mathbf{k}\mathbf{e}(\mathbf{k}, \alpha)) (\mathbf{j}\mathbf{e}(\mathbf{k}, \alpha))}{\omega^2 - \omega_0^2(\mathbf{f}, \alpha)} - \mathbf{j}, \mathbf{k} \right]^2, & \nu = 1, 2, 3, \end{cases} \quad (3.16a)$$

$$\left| \sum_{n \neq 0} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) e^{i\mathbf{k}\mathbf{r}_n} \right|^2 = \begin{cases} \frac{N}{2} \sum_{\delta=1}^{\vartheta/2} [S_{\nu\delta}(p)]^2 (\mathbf{k}\mathbf{e}_{p\delta})^2 (\delta_{\mathbf{f}_{p\delta}, \mathbf{k}} + \delta_{\mathbf{f}_{p\delta}, -\mathbf{k}}), & \nu = 4, \dots, \vartheta/2, \end{cases} \quad (3.16b)$$

$$\left| \sum_{n \neq 0} (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n)) e^{i\mathbf{k}\mathbf{r}_n} \right|^2 = \begin{cases} \frac{N}{2} (\mathbf{k}\mathbf{e}_{p\nu})^2 (\delta_{\mathbf{f}_{p\nu}, \mathbf{k}} + \delta_{\mathbf{f}_{p\nu}, -\mathbf{k}}), & \nu = \vartheta/2 + 1, \dots, \vartheta; \end{cases} \quad (3.16c)$$

$$\sum_n (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2 = \begin{cases} \frac{d \ln \omega^2}{d\varepsilon} \left[\frac{2\varepsilon \omega^4}{N} \sum_{i\alpha\alpha'} \frac{(\mathbf{k}\mathbf{e})(\mathbf{j}\mathbf{e})(\mathbf{k}\mathbf{e}')(\mathbf{j}\mathbf{e}')}{(\omega^2 - \omega_0^2)(\omega^2 - \omega_0'^2)} - \varepsilon (\mathbf{j}, \mathbf{k})^2 \right], & \nu = 1, 2, 3, \end{cases} \quad (3.17a)$$

$$\sum_n (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2 = \begin{cases} \sum_{\delta=1}^{\vartheta/2} [S_{\nu\delta}(p)]^2 (\mathbf{k}\mathbf{e}_{p\delta})^2, & \nu = 4, \dots, \vartheta/2, \end{cases} \quad (3.17b)$$

$$\sum_n (\mathbf{k} \Phi_{p\nu}(\mathbf{r}_n))^2 = \begin{cases} (\mathbf{k}\mathbf{e}_{p\nu})^2, & \nu = \vartheta/2 + 1, \dots, \vartheta. \end{cases} \quad (3.17c)$$

We have used the abbreviated notation

$$\omega^2 = \omega_{p\nu}^2, \quad \mathbf{e} = \mathbf{e}(\mathbf{f}, \alpha), \quad \mathbf{e}' = \mathbf{e}(\mathbf{f}, \alpha'), \quad \omega_0^2 = \omega_0^2(\mathbf{f}, \alpha), \quad \omega_0'^2 = \omega_0^2(\mathbf{f}, \alpha').$$

We go from (3.13) to the differential scattering cross section:

$$\frac{d^2\sigma(\mathbf{k}, \Delta E)}{dE d\Omega} = \frac{\mu^2 k_2}{4\pi^2 \hbar^4 k_1} \sum_{p \nu=1}^{\vartheta_p} \overline{M}_{p\nu} \delta(\Delta E \mp \hbar\omega_{p\nu}). \quad (3.18)$$

Here μ is the mass of the neutron, $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors of the neutron before and after scattering,

$$\Delta E = E_1 - E = \hbar^2 (k_1^2 - k_2^2) / 2\mu. \quad (3.19)$$

In measuring the differential cross section (3.18) there is an unavoidable averaging over a small energy interval which nevertheless contains a macroscopic number of frequencies; the weighting function $\gamma(\Delta E - \Delta E')$ is determined by the experimental resolution ($\int \gamma(\Delta E - \Delta E') d\Delta E' = 1$). We see that in this case $\gamma(\Delta E \mp \hbar\omega_{p\nu})$ appears in the expression (3.18) for the cross section in place of the δ function. In particular the transition from summation over the quasicontinuous spectrum to integration is done immediately.

In order to solve the inverse problem, it is necessary to separate out the part of the scattering cross section which is related to the scattering by the impurity nucleus and by the disturbed region in its vicinity. We look at the sum over ν from 4 to $\mathcal{J}/2$ in (3.18). Only the terms (3.16b) and (3.17b) contribute to it. Remembering that for an orthogonal matrix

$$\sum_{\nu=1}^{\mathcal{J}/2} [S_{\nu\delta}(p)]^2 = 1, \quad (3.20)$$

and also relations (2.10) and (2.14), we have

$$\begin{aligned} \frac{N}{2} \sum_{\nu=4}^{\mathcal{J}/2} \sum_{\delta=1}^{\mathcal{J}/2} [S_{\nu\delta}(p)]^2 (\mathbf{k} \mathbf{e}_{p\delta})^2 (\delta_{f_{p\delta}, \mathbf{k}} + \delta_{f_{p\delta}, -\mathbf{k}}) \\ = \frac{N}{2} \sum_{\delta=1}^{\mathcal{J}/2} (\mathbf{k} \mathbf{e}_{p\delta})^2 (\delta_{f_{p\delta}, \mathbf{k}} + \delta_{f_{p\delta}, -\mathbf{k}}) \\ - \frac{N}{2} \sum_{\delta=1}^{\mathcal{J}/2} (\mathbf{k} \mathbf{e}_{p\delta})^2 \sum_{\nu=1}^3 \eta_{\nu}^{-1}(p) (\mathbf{j}_{\nu} \mathbf{e}_{p\delta})^2 (\delta_{f_{p\delta}, \mathbf{k}} + \delta_{f_{p\delta}, -\mathbf{k}}); \end{aligned} \quad (3.21)$$

$$\begin{aligned} \sum_{\nu=4}^{\mathcal{J}/2} \sum_{\delta=1}^{\mathcal{J}/2} [S_{\nu\delta}(p)]^2 (\mathbf{k} \mathbf{e}_{p\delta})^2 = \sum_{\delta=1}^{\mathcal{J}/2} (\mathbf{k} \mathbf{e}_{p\delta})^2 \\ - \sum_{\delta=1}^{\mathcal{J}/2} (\mathbf{k} \mathbf{e}_{p\delta})^2 \sum_{\nu=1}^3 \eta_{\nu}^{-1}(p) (\mathbf{j}_{\nu} \mathbf{e}_{p\delta})^2. \end{aligned} \quad (3.22)$$

The first term in (3.21) and (3.16c) leads to the usual expression for the coherent scattering cross section

$$\begin{aligned} \frac{d^2\sigma_{\text{coh}}^{(0)}(\mathbf{k}, \Delta E)}{dE d\Omega} = \frac{\mu^2 \bar{A}^2 R k_2 e^{-w}}{4\pi^2 \hbar^4 k_1} N \sum_{\alpha} (\mathbf{x} \mathbf{e}(\mathbf{k}, \alpha))^2 \\ \times \frac{1}{\hbar\omega_0(\mathbf{k}, \alpha)} \delta(\Delta E \mp \hbar\omega_0(\mathbf{k}, \alpha)), \end{aligned} \quad (3.23)$$

while the first term in (3.22) and (3.17c) gives the usual expression for the incoherent scattering cross section

$$\begin{aligned} \frac{d^2\sigma_{\text{inc}}^{(0)}(\mathbf{k}, \Delta E)}{dE d\Omega} = \frac{\mu^2 C R k_2 e^{-w}}{2\pi^2 \hbar^4 k_1} \sum_{\mathbf{f}\alpha} (\mathbf{x} \mathbf{e}(\mathbf{f}, \alpha))^2 \\ \times \frac{1}{\hbar\omega_0(\mathbf{f}, \alpha)} \delta(\Delta E \mp \hbar\omega_0(\mathbf{f}, \alpha)) \end{aligned} \quad (3.24)$$

of the original ideal lattice. (In deriving (3.23) one must consider that of the two vectors $\mathbf{k}$ and $-\mathbf{k}$ only one can belong to the region of summation over $\mathbf{f}$, corresponding to half the fundamental cell of the reciprocal lattice.)

Now we separate the coherent and incoherent scattering. To do this we use the idea developed in [8]: we assume that the differential cross section (3.21) is measured as a function of ΔE for fixed momentum transfer $\mathbf{k} = 2\pi\mathbf{b}$ (where $\mathbf{b}$ is a vector of the reciprocal lattice). Then for $\Delta E \neq 0$ the quantity $d^2\sigma_{\text{coh}}^{(0)}(\mathbf{k} = 2\pi\mathbf{b}, \Delta E)/dE d\Omega$ is zero since $\omega_0(2\pi\mathbf{b}, \alpha) = 0$. At the same time the contribution to the cross section of the second term in (3.21) vanishes.

After substituting $\mathbf{k} = 2\pi\mathbf{b}$, the expressions (3.15) and (3.16a) give

$$(D/m) (\mathbf{j}, \mathbf{k})^2 d \ln \omega_{p\nu}^2 / d\varepsilon, \quad (3.25)$$

$$D = [\bar{A}(1 - \varepsilon) - A_0]^2 + 1/4 B_0^2 K_0 (K_0 + 1) - (1 - \varepsilon) C. \quad (3.26)$$

It is easy to see that if we neglect the anisotropy of the Debye-Waller factor, the cross section (3.18) for $\mathbf{k} = 2\pi\mathbf{b}$ can be represented in the following form ($\Delta E \neq 0$):

$$d^2\sigma(\mathbf{k} = 2\pi\mathbf{b}, \Delta E)/dE d\Omega = R \mathbf{x}^{\mathbf{k}} \mathbf{x}^{\mathbf{k}} T^{\mathbf{k}}(\Delta E), \quad (3.27)$$

where $T^{\mathbf{k}}(\Delta E)$ is a tensor with the symmetry of the crystal. By measuring the cross section in the general case for three values $\mathbf{k} = 2\pi\mathbf{b}$ (cf. [8]) which are not coplanar, we can find the principal values of the tensor $T^{\mathbf{k}}(\Delta E)$ and consequently $\text{Sp } T(\Delta E)$.

For the case of uniaxial crystals it is sufficient to measure the cross section for just two different values $\mathbf{k} = 2\pi\mathbf{b}$, while for the cubic crystal only one value is needed.

If the coherent scattering is small ($\bar{A} \approx 0$), the need for fixing $\mathbf{k} = 2\pi\mathbf{b}$ disappears. Measurement of the total cross section for three orientations of the single crystal which differ by cyclic permutation of the axes (cf. [8, 9]) or for a polycrystal, again gives the value of $\text{Sp } T(\Delta E)$.

Thus we shall assume that the value of $\text{Sp } T(\Delta E)$ is known and, on this basis we shall treat the contribution to the cross section of the remaining terms (3.17a) and the second term in (3.22). Considering (2.12), (2.13), and (2.18), to within a multiplicative factor the contribution of these terms to the cross section (3.18) is given by the expression (including the average over a small energy interval)

$$\sum_p \sum_{\nu=1}^3 \frac{\bar{n}(\omega_{p\nu}) + 1/2 \pm 1/2}{\omega_{p\nu}} \gamma(\Delta E \mp \hbar\omega_{p\nu}) - 3 \sum_p \frac{\bar{n}(\omega_{0p}) + 1/2 \pm 1/2}{\omega_{0p}} \gamma(\Delta E \mp \hbar\omega_{0p}). \quad (3.28)$$

We note that for the special case of a cubic crystal the three frequencies $\omega_{p\nu}$, $\nu = 1, 2, 3$, coincide. Keeping this in mind and also the relation (2.16), we immediately arrive at an expression of the type (3.28) after summation of (3.17a) over ν .

Using (2.25), the transformation (3.28) gives

$$\sum_p \sum_{\nu=1}^3 \delta\omega_{0p}^2 z_{p\nu} \frac{\partial}{\partial\omega_{0p}^2} \left\{ \frac{\bar{n}(\omega_{0p}) + 1/2 \pm 1/2}{\omega_{0p}} \gamma(\Delta E \mp \hbar\omega_{0p}) \right\}.$$

Changing from summation over p to an integration, and then integrating by parts, we finally get the expression

$$- \sum_{\nu=1}^3 \int d\omega^2 \frac{\partial z_{\nu}(\omega^2)}{\partial\omega^2} \frac{\bar{n}(\omega) + 1/2 \pm 1/2}{\omega} \gamma(\Delta E \mp \hbar\omega). \quad (3.29)$$

Suppose that the relative concentration of impurity atoms is η . Then going to the cross section and taking it as usual per atom of the crystal, if we assume good resolution we get an expression of the form (3.27) with the following value of $\text{Sp } T(\Delta E)$ [cf. (3.25) and (3.29)]:

$$\text{Sp } T(\Delta E) = \text{Sp } T_0(\Delta E) + \eta \text{Sp } T_1(\Delta E), \quad (3.30)$$

where $\text{Sp } T_0(\Delta E)$ is determined in an obvious way from (3.24), while

$$\text{Sp } T_1(\Delta E) = \frac{\mu^2 k_2 e^{-w}}{2\pi^2 \hbar^6 k_1} \sum_{\nu=1}^3 \left[D f_{\nu}(\omega^2) - C \frac{\partial z_{\nu}(\omega^2)}{\partial\omega^2} \right] \times \left[\bar{n}(\omega) + \frac{1}{2} \pm \frac{1}{2} \right] \Big|_{\omega=|\Delta E|/\hbar} \quad (3.31)$$

The last expression is obviously entirely due to the scattering from impurity atoms. [In deriving (3.31) we have used the relation (2.27).]

In the case of a cubic crystal there is no need for three independent measurements, and we get directly

$$\frac{d^2\sigma^{(1)}(\mathbf{k} = 2\pi\mathbf{b}, \Delta E)}{dE d\Omega} = \frac{R}{3} \eta \text{Sp } T_1(\Delta E) = \frac{R\mu^2 k_2 e^{-w}}{2\pi^2 \hbar^6 k_1} \eta \left[D f(\omega^2) - C \frac{\partial z(\omega^2)}{\partial\omega^2} \right] \left[\bar{n}(\omega) + \frac{1}{2} \pm \frac{1}{2} \right] \Big|_{\omega=|\Delta E|/\hbar}. \quad (3.31')$$

As already mentioned, if the coherent scattering is small (3.31) and (3.31') hold for arbitrary $\mathbf{k}$. Let $\epsilon = 0$. Then

$$D = (\bar{A} - A_0)^2 + 1/4 B_0^2 K_0 (K_0 + 1) - C; \quad \partial z_{\nu}(\omega^2) / \partial\omega^2 = 0, \quad f_{\nu}(\omega^2) = g^{(\nu)}(\omega^2). \quad (3.32)$$

If the selected atom belongs to the isotopic mixture

of the regular lattice, we find, by averaging (3.32), that $D = 0$ and expression (3.31) vanishes.

If $C \ll D$, the need for calculating the contribution to (3.18) from (3.17a) and the second term in (3.22) disappears; the same remark applies to the averaging of the polarization vectors which is done only to simplify the final formulas. In this case we find from (3.25) and (3.26) that the cross section due to the impurity atoms is

$$\frac{d\sigma^{(1)}(\mathbf{k} = 2\pi\mathbf{b}, \Delta E)}{dE d\Omega} = R\eta \chi^i \chi^k T_1^{ik}(\Delta E), \quad (3.33)$$

$$T_1^{ik}(\Delta E) = \frac{\mu^2 k_2 e^{-w} D}{2\pi^2 \hbar^6 k_1} \left[\sum_{\nu=1}^3 j_{\nu}^i j_{\nu}^k f_{\nu}(\omega^2) \right] \left[\bar{n}(\omega) + \frac{1}{2} \pm \frac{1}{2} \right] \Big|_{\omega=|\Delta E|/\hbar}, \quad (3.34)$$

$$D = [\bar{A} (1 - \epsilon) - A_0]^2 + 1/4 B_0^2 K_0 (K_0 + 1). \quad (3.35)$$

We see that the tensor $T_1^{ik}(\Delta E)$ has a very simple form, directly accessible to analysis, while its principal values, which are again found by making the required number of independent measurements (three, two, or one, depending on the symmetry of the crystal), are equal to

$$T_1^{\nu\nu}(\Delta E) = \frac{\mu^2 k_2 e^{-w} D}{2\pi^2 \hbar^6 k_1} f_{\nu}(\omega^2) \left[\bar{n}(\omega) + \frac{1}{2} \pm \frac{1}{2} \right] \Big|_{\omega=|\Delta E|/\hbar}. \quad (3.36)$$

This case is of the greatest interest for the analysis of the vibration spectrum of the crystal.

4. SCATTERING OF SLOW NEUTRONS WHEN DISCRETE FREQUENCIES ARE PRESENT

When condition (2.24) is satisfied, there are discrete frequencies in addition to the quasicontinuous spectrum. Although the number of such frequencies is small, their relative importance for the scattering cross section corresponding to the impurity atoms is considerable, while in some cases they may play a decisive role. This is related to the fact that, because of their spatial localization, the discrete frequencies correspond to larger displacement amplitudes of the impurity atom, comparable with or exceeding the total amplitude of its vibration in all the frequencies of the quasicontinuous spectrum.

Formula (3.12), which was derived using only Eq. (2.1) and the orthogonality and completeness of its solutions, and all the remarks concerning the Debye-Waller factor which were made in the preceding section, are also valid here. The change is the appearance of a factor in the form of a Bessel function, since the displacement amplitude in a localized vibration does not have the factor $\sim N^{-1/2}$.

It should be mentioned that when there are localized normal modes the question of strict degeneracy of such discrete frequencies plays a special role (cf. [4]). In calculating the differential cross

section we are faced with the necessity of summing over all final states with a fixed energy, corresponding to different occupation numbers of the localized normal oscillators, i.e., we must consider multiquantum processes for such oscillators.

In the case of scattering with $|\Delta E| \leq \hbar\omega_0 \max$, i.e., when only phonons of the quasicontinuous spectrum are excited, we arrive at an expression for the square modulus of the matrix element which differs from (3.5) by the substitution

$$A_n A_{n'} \rightarrow A_n A_{n'} \prod_{\lambda=1}^3 I_0(X_{nn'}^{(\lambda)}), \quad B_n^2 \rightarrow B_n^2 \prod_{\lambda=1}^3 I_0(X_{nn}^{(\lambda)}), \quad (4.1)$$

where for brevity we have introduced the notation

$$X_{nn'}^{(\lambda)} = \sum_{(\lambda)} \frac{\hbar (\mathbf{k} \Phi_{D\lambda}(\mathbf{r}_n)) (\mathbf{k} \Phi_{D\lambda}(\mathbf{r}_{n'}))}{2\omega_{D\lambda} \sqrt{m_n m_{n'}}} \operatorname{sh}^{-1} \frac{\hbar\omega_{D\lambda}}{2T}. \quad (4.2)$$

The symbol $\prod_{\lambda}'$ denotes a product over the various discrete frequencies, $\sum_{(\lambda)}$ is a summation over

all the normal vibrations corresponding to a given discrete frequency $\omega_{D\lambda}$. In deriving (4.1)–(4.2), we have used the following summation formula for Bessel functions:

$$\sum_{n_1=-\infty}^{\infty} \dots \sum_{n_s=-\infty}^{\infty} \prod_{i=1}^s I_{n_i}(x_i) = I_m \left(\sum_{i=1}^s x_i \right),$$

$$\sum_{i=1}^s n_i = m$$

Similarly by using (3.2) we can write an expression corresponding to creation or absorption of a quantum of the discrete frequency $\omega_{D\nu}(|\Delta E|) = \hbar\omega_{D\nu}$:

$$M_{D(\nu)} = \sum_n \sum_{n'} A_n A_{n'} \exp \{ i\mathbf{k}(\mathbf{r}_n - \mathbf{r}_{n'}) \} \\ \times \exp \left(-\frac{\omega_n + \omega_{n'}}{2} \right) \exp \left(\pm \frac{\hbar\omega_{D\nu}}{2T} \right) I_1(X_{nn'}^{(\nu)}) \prod_{\lambda \neq \nu}' I_0(X_{nn'}^{(\lambda)}), \\ + \frac{1}{4} \sum_n B_n^2 K_n(K_n + 1) e^{-\omega_n} \exp \left(\pm \frac{\hbar\omega_{D\nu}}{2T} \right) \\ \times I_1(X_{nn}^{(\nu)}) \prod_{\lambda \neq \nu}' I_0(X_{nn}^{(\lambda)}). \quad (4.3)$$

But it is easy to see that for $T \ll \hbar\omega_0 \max$ the arguments of the Bessel functions in (4.2) are much less than unity, and consequently it is again meaningful to expand as in (3.4); also

$$I_0 \approx 1.$$

On the other hand assuming that $R'/\hbar\omega_0 \max < 1$, these expansions can be used over the whole temperature range in which the one-quantum approximation is valid. Considering this region, we find that all the results of the preceding section for the

case of excitation of quanta of the quasicontinuous spectrum ($|\Delta E| \leq \hbar\omega_0 \max$) also remain valid in the presence of discrete frequencies.

We note that in the limiting case of $m_0/m \ll 1$, which corresponds to a marked localization of the normal vibrations for the discrete frequencies, it is useful to keep the true value for w_0 . In other words, in this case when using the formulas of the preceding section it is sensible to make the replacement

$$A_0^2 e^{-w} \rightarrow A_0^2 e^{-w_0} \prod_{\nu=1}^3 I_0(X_{00}^{(\nu)}), \quad B_0^2 e^{-w} \rightarrow B_0^2 e^{-w_0} \prod_{\nu=1}^3 I_0(X_{00}^{(\nu)}), \quad (4.4)$$

leaving the other terms unchanged [including those linear in A — cf. (4.1)–(4.2)]. Obviously because of (2.30) and (4.2),

$$X_{00}^{(\nu)} = \sum_{(\nu)} \frac{R(\mathbf{x} \mathbf{j}_{\nu})^2}{\hbar\omega_{D\nu}} \frac{d \ln \omega_{D\nu}^2}{d\varepsilon} \operatorname{sh}^{-1} \frac{\hbar\omega_{D\nu}}{2T}. \quad (4.5)$$

Now we determine the cross section for scattering with excitation of a discrete frequency. In view of the above remarks, the derivation is analogous to that in the previous section for the quasicontinuous spectrum. The difference is that the only contributions to the cross section (3.18) in this case come from (3.15), (3.16), and (3.17a), where $\omega^2 = \omega_{D\nu}^2$. The result is

$$\frac{d^2\sigma(\mathbf{k}, \Delta E)}{dE d\Omega} = \frac{\mu^2 k_2 R e^{-w}}{4\pi^2 \hbar^4 k_1} \eta \sum_{\nu=1}^3 \frac{1}{\hbar\omega_{D\nu}} \frac{d \ln \omega_{D\nu}^2}{d\varepsilon} \left\{ \overline{A} \left(\mathbf{j}_{\nu} \mathbf{x} - \varepsilon \omega_{D\nu}^2 \right. \right. \\ \times \sum_{\alpha} \frac{(\mathbf{x} \mathbf{e}(\mathbf{k}, \alpha)) (\mathbf{j}_{\nu} \mathbf{e}(\mathbf{k}, \alpha))}{\omega_{D\nu}^2 - \omega_0^2(\mathbf{k}, \alpha)} - A_0(\mathbf{j}_{\nu} \mathbf{x}) \left. \right\}^2 + \left[\frac{1}{4} B_0^2 K_0(K_0 + 1) \right. \\ \left. - (1 - \varepsilon) C \right] (\mathbf{j}_{\nu} \mathbf{x})^2 + C \left[\frac{2\varepsilon^2 \omega_{D\nu}^4}{N} \sum_{\mathbf{i} \alpha \alpha'} \frac{(\mathbf{x} \mathbf{e})(\mathbf{j}_{\nu} \mathbf{e})(\mathbf{x} \mathbf{e}')(\mathbf{j}_{\nu} \mathbf{e}')}{(\omega^2 - \omega_0^2)(\omega^2 - \omega_0'^2)} \right. \\ \left. - \varepsilon(\mathbf{j}_{\nu} \mathbf{x})^2 \right] \left(\overline{n}_{D\nu} + \frac{1}{2} \pm \frac{1}{2} \right) \delta(\Delta E \mp \hbar\omega_{D\nu}). \quad (4.6)$$

Expression (4.6) actually determines the total cross section for scattering of neutrons by the lattice ($|\Delta E| > \hbar\omega_0 \max$), excluding the multiquantum background, which we are neglecting (cf. below). There is therefore no need for additional conditions ($\mathbf{k} = 2\pi\mathbf{b}$) for eliminating the coherent scattering. Nevertheless, to simplify (4.2) we shall fix $\mathbf{k} = 2\pi\mathbf{b}$, or assume that $\mathbf{k}$ is arbitrary, but sufficiently small so that

$$\omega_0^2(\mathbf{k}, \alpha) \ll \omega_{D\nu}^2. \quad (4.7)$$

As a result we get

$$\operatorname{Sp} T(\Delta E) = \frac{\mu^2 k_2 e^{-w}}{4\pi^2 \hbar^4 k_1} \sum_{\nu=1}^3 \frac{1}{\omega_{D\nu}} \left(D \frac{d \ln \omega_{D\nu}^2}{d\varepsilon} + C \right) \\ \times \left(\overline{n}_{D\nu} + \frac{1}{2} \pm \frac{1}{2} \right) \delta(\Delta E \mp \hbar\omega_{D\nu}) \quad (4.8)$$

for the case of a crystal of arbitrary symmetry, and

$$\frac{d^2\sigma(k, \Delta E)}{dE d\Omega} = \eta \frac{\mu^2 R k_2 e^{-w}}{4\pi^2 \hbar^5 k_1 \omega_D} \left(D \frac{d \ln \omega_D^2}{d\epsilon} + C \right) \times \left(\bar{n}_D + \frac{1}{2} \pm \frac{1}{2} \right) \delta(\Delta E \mp \hbar \omega_D) \quad (4.9)$$

for the case of cubic symmetry. The value of the quantity $d \ln \omega_{D\nu}^2 / d\epsilon$ is determined by (2.32) or by the expression (2.33), using (2.20).

The procedure for obtaining (4.8) is the same as in the preceding section—one uses a polycrystal [with $\bar{A} \approx 0$ and arbitrary k , or with arbitrary $\bar{A}$ and k satisfying (4.7)], or measures the cross section for the appropriate number of independent values of k (equal to $2\pi b$ or satisfying (4.7)) using a single crystal.

The excited states of the discrete normal oscillators exist for a finite time even when $T = 0$. Introducing a width for the excited state $\Gamma_{D\nu}$, and approximating the instrument resolution curve $\gamma(\Delta E - \Delta E')$ (cf. the preceding section) by a function of Lorentz type with characteristic parameter Γ , we get expressions which differ from (4.6), (4.8)–(4.9) by the replacement

$$\delta(\Delta E \mp \hbar \omega_{D\nu}) \rightarrow \frac{1}{\pi} \frac{(\Gamma_{D\nu} + \Gamma)/2}{(\Delta E \mp \hbar \omega_{D\nu})^2 + (\Gamma_{D\nu} + \Gamma)^2/4} \quad (4.10)$$

As direct calculations show, the ratio $\Gamma_D / \hbar \omega_D$ for $T \ll \Theta$ is of order 10^{-2} , while it is even much less for $\omega_0 > 2\omega_{0\max}$. Thus for sufficiently good resolution, when $\Delta E = \hbar \omega_{D\nu}$ one should see sharp peaks. Obviously from the area under the peaks one can determine the corresponding values of $d \ln \omega_{D\nu}^2 / d\epsilon$.

When $\Delta E > \hbar \omega_{0\max}$, in addition to one-phonon scattering corresponding to the discrete levels, one will also see two-phonon scattering. The possibility of observing neutron scattering with excitation of discrete levels is obviously essentially connected with the assumption that the two-phonon scattering is small. Let us estimate the cross section for such a process.

Let the one-phonon scattering from the ideal host be primarily coherent. Then, computing the cross section for the two-phonon process for $k = 2\pi b$, we find

$$\frac{d^2\sigma_2(k = 2\pi b, \Delta E)}{dE d\Omega} \sim \frac{\mu^2 A_2^2 k_2}{\pi^2 \hbar^5 k_1} e^{-w} \left(\frac{R}{\Delta E} \right)^2 \psi \left(\frac{\Delta E}{\hbar} \right) \left[\bar{n} \left(\frac{\Delta E}{\hbar} \right) + 1 \right], \quad \Delta E > 0, \quad (4.11)$$

where $\psi(\omega)$ is the normalized distribution function for the frequencies in the phonon spectrum.

Analysis of (4.11) shows that the cross section for two-phonon scattering, compared to R times (4.8), (4.10) [or (4.9), (4.10)] contains the small parameter $R/\Delta E$, while for sufficiently good resolution it contains another small parameter $\Gamma_{D\nu} / \hbar \omega_{0\max}$. Thus by suitably choosing the parameters of the impurity nucleus and limiting the temperature one can comparatively easily guarantee conditions in which one will observe sharp peaks in the cross section corresponding to discrete levels, with a low concentration of impurity atoms.

We note that for a large mass difference ($m_0/m \ll 1$) the values of the cross section for two-quantum excitations of localized oscillators also become significant. Thus in addition to the maxima in the scattering cross section for $\Delta E = \hbar \omega_{D\nu}$, satellites should appear at $\Delta E = 2\hbar \omega_{D\nu}$, etc. ($\Delta E > 0$).

5. DISCUSSION OF RESULTS

The results obtained in the preceding sections allow us to analyze what information concerning the vibration spectrum can be gotten from measurements of the cross section for scattering of slow neutrons by crystals containing isolated impurity atoms.

The decisive point for the inverse problem is the possibility of separating out the actual cross section for scattering by the impurity atom and the disturbed region which is produced around it. The problem is solved most simply if one can use impurity nuclei whose scattering is anomalously large in the energy region of interest. Several isotopes with low-lying resonance levels are known at present. But the energies of all these levels are of the order of one eV, so that the use of these isotopes requires the development of a special experimental technique.

If, however, the total scattering from the impurity nuclei does not predominate, the problem of separating out $d^2\sigma^{(1)}/dE d\Omega$ is more complicated. To isolate the coherent scattering one must measure the scattering cross section of single crystals as a function of ΔE for fixed momentum transfer $k = 2\pi b$. If the coherent scattering is negligibly small, the measurement can be made on a polycrystal, where a complete averaging of the polarization occurs (cf. [8] and also Sec. 3). Thus in both cases we at least get the quantity (3.30), and

if $d^2\sigma^{(1)}/dE d\Omega$ is comparable to $d^2\sigma_{inc}^{(0)}/dE d\Omega$, the quantity $\eta \text{ Sp } T_1(\Delta E)$, which is what we want, can be separated out.

But the most interesting question is whether, from experiments on scattering of neutrons by crystals with low impurity concentrations, one can establish the distribution function for the squared frequencies of the initial ideal lattice, (2.22).

The possibility of separating $d^2\sigma^{(1)}/dE d\Omega$ when $\eta \ll 1$ definitely assumes that

$$D \gg C. \quad (5.1)$$

In this case, as shown in Sec. 3, measuring the cross section for three noncoplanar values of $k = 2\pi b$ (for uniaxial crystals two independent measurements are sufficient, for a cubic crystal, one) for a fixed value $\Delta E \neq 0$, one can find the principal values of the tensor $T^{ik}(\Delta E)$ of (3.36) and consequently the quantities $f_\nu(\omega^2)$. The existence of the functional relation (2.28) between $f_\nu(\omega^2)$ and $g^{(\nu)}(\omega^2)$ then allows us to get $g^{(\nu)}(\omega^2)$, and thus we can affirm that it is possible to determine $g(\omega^2)$. (To calculate the integral appearing in (3.28) in first approximation one can use some simple model, and then use the method of successive approximations.)

One should remember that the whole theory is based on the assumption that the force constants are unchanged by the introduction of the foreign atom. But when the masses m_0 and m are markedly different, one can apparently neglect changes in the force constants (cf, also [4]), so that this assumption becomes completely reasonable.

A very important point is the possibility of detecting from experiments on scattering the presence of discrete frequencies and to determine their magnitude as well as the quantity $d \ln \omega_{D\nu}^2/d\epsilon$. Agreement between the experimental values of $\omega_{D\nu}$ and those which are computed from the solution of (2.20), or, for a cubic crystal, are given by the equation

$$\epsilon \omega_D^2 \int_0^{\omega_{0max}^2} \frac{g(\omega^2) d\omega^2}{\omega_D^2 - \omega^2} = 1, \quad (5.2)$$

would serve as a good confirmation of the correctness of the initial assumption of unchanged values of the force constants. In solving (2.20) we can again choose $g^{(\nu)}(\omega^2)$ from some crude model. If $\omega_{D\nu}$ is sufficiently high, to find it (as shown in [3,4]) it is sufficient to know only the first moments of the function $g^{(\nu)}(\omega^2)$, which can be determined to good accuracy by an independent method. In particular, in the case of a cubic crys-

tal the first moment $\langle \omega^2 \rangle$ can be found from the deviation of the specific heat from its classical limit at high temperatures.

We remark that there is particular interest in determining $g(\omega^2)$ for a host with very weak one-phonon incoherent scattering.

If Eq. (5.1) is not satisfied, we must use for D the general expression in the form (3.26). From the form of this expression it follows that one can get a marked increase in $d^2\sigma^{(1)}/dE d\Omega$ by using heavy impurity atoms ($m_0/m \gg 1$), or by using atoms for which A_0 has the opposite sign from A .

As shown in earlier papers of the authors, [3,4] when $m_0/m \gg 1$ the quantity f_ν (2.28) has a strong resonant character:

$$f_\nu(x) = \frac{1}{\pi |\epsilon|} \sqrt{\frac{x}{x_\nu}} \frac{\lambda_\nu}{(x - x_\nu)^2 + \lambda_\nu^2}, \quad (5.3)$$

where

$$x = \frac{\omega^2}{\omega_{0max}^2}, \quad x_\nu = \frac{1}{|\epsilon| \langle x^{-1} \rangle_\nu}, \quad \lambda_\nu = \frac{\pi d_\nu}{|\epsilon|^{1/2} \langle x^{-1} \rangle_\nu^{1/2}}. \quad (5.4)$$

The symbol $\langle \dots \rangle_\nu$ denotes an average with weight $g^{(\nu)}$ (2.21), so that

$$\langle x^{-1} \rangle_\nu \equiv \int_0^1 \frac{g^{(\nu)}(x) dx}{x};$$

d_ν is a constant determined from the low frequency limit ($x \ll 1$) of the function $g^{(\nu)}(x)$, which has the form

$$g^{(\nu)}(x) = d_\nu \sqrt{x}.$$

Because of this, the cross section for one-quantum excitations (3.33)–(3.34) will be localized in a narrow energy interval around

$$\Delta E = \frac{\hbar \omega_{0max}}{|\epsilon|^{1/2} \langle x^{-1} \rangle_\nu^{1/2}}, \quad (5.5)$$

while the intensity at the maximum will be $\sim 1/\lambda_\nu$ times the usual scattering cross section from the impurity centers. Apparently this case is very favorable for detecting the one-quantum scattering of neutrons from impurity atoms when their concentration is low.

We remark that in the case considered the two-quantum (multiquantum) excitations may become important, which will correspond to peaks in the neutron scattering cross section for

$$\Delta E = \frac{2\hbar \omega_{0max}}{|\epsilon|^{1/2} \langle x^{-1} \rangle_\nu^{1/2}} \quad \left(\Delta E = \frac{n\hbar \omega_{0max}}{|\epsilon|^{1/2} \langle x^{-1} \rangle_\nu^{1/2}} \right).$$

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LEPTON γ_5 SYMMETRY IN WEAK INTERACTIONS AND IN $\Delta S = -\Delta Q$ DECAY MODES

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The assumptions of an additive conservation law of leptonic charge and of definite chiralities of the electron and muon in weak interactions determine uniquely the lepton pairs (ν_e, e^-) and (ν_μ, μ^-) in the four-fermion interaction with baryons only in the case of V-coupling. The possible violation of V-coupling in the $\Delta S = -\Delta Q$ decay modes of strange particles, and hence the possible participation of lepton pairs of the forms ($\tilde{\nu}_\mu, e^-$) and ($\tilde{\nu}_e, \mu^-$) in weak interactions, is discussed.

THE recent experiment at Brookhaven^[1] which has disclosed an asymmetry of electrons and muons with respect to neutrinos originating from pion decays, is a brilliant experimental confirmation of the two-neutrino hypothesis which has been discussed in recent years^[2-7] (further references are given in [1]).

In a previous paper^[5] the author has obtained the two neutrinos from a classification of neutral massless fermions with respect to the sign of the leptonic charge and chirality (the eigenvalues of the operator γ_5)¹⁾:

$$\begin{array}{cccc} \nu_e & \nu_\mu & \tilde{\nu}_e & \tilde{\nu}_\mu \\ l & +1 & -1 & -1 & +1 \\ \gamma_5 & +1 & +1 & -1 & -1 \end{array} \quad (1)$$

In conformity with the work of Marshak and Sudarshan^[8] it has been assumed in [5] that the massive leptons (e and μ) also possess definite chiralities in weak interactions, in the sense that each occurs in the weak interaction Lagrangian always with the same factor $(1 + \gamma_5)$ or $(1 - \gamma_5)$ and that the Lagrangian is invariant with respect to the following transformations²⁾:

$$e^\pm \rightarrow \pm \gamma_5 e^\pm, \quad \mu^\pm \rightarrow \pm \gamma_5 \mu^\pm. \quad (2)$$

In the case of V-coupling the γ_5 symmetry implies conservation of the lepton chirality in weak interactions, introducing a new additive quantum

number in addition to electric and leptonic charge. If one now attributes different leptonic charges to electrons and muons of equal electric charge (following the proposals of Konopinski and Mahmoud^[10] and Zel'dovich^[11]), i.e.,^[5]

$$\begin{array}{cccc} e^- & \mu^- & e^+ & \mu^+ \\ l & +1 & -1 & -1 & +1 \\ Q & -1 & -1 & +1 & +1 \end{array} \quad (3)$$

then the conservation of the electric and leptonic charges and chirality will imply uniquely that only lepton pairs of the form

$$L_1 = (\nu_e, e^-), \quad L_2 = (\nu_\mu, \mu^-) \quad (4)$$

must participate in all weak interactions (we do not consider here neutral currents). In particular, the form of the muon decay interaction will be uniquely determined as:

$$\mu^- \rightarrow e^- + \tilde{\nu}_e + \nu_\mu \quad (5)$$

and processes of the form

$$K^0 \rightarrow \mu^+ + e^-, \quad e^- + 2p \rightarrow \mu^+ + 2n, \quad (6)$$

will be forbidden, although they conserve leptonic charge alone.

However it is important to note that owing to the definition (3) of leptonic charge, the conservation laws of the charges and γ_5 -invariance determine uniquely both the muon decay interaction (5) and the form of the four-fermion muon decay interaction, independently of the assumption of V-coupling. Indeed, using the Fierz identities, it is easy to see that the two expressions of the four-fermion interaction Hamiltonian which are allowed by the conservation of the charges and by γ_5 -invariance

$$H_V \sim (\bar{\nu}_\mu \gamma_\alpha (1 + \gamma_5) \mu^-) (\bar{e}^- \gamma_\alpha (1 + \gamma_5) \nu_e), \quad (7)$$

¹⁾Cf. also the papers by Schwinger,^[2] Nishijima^[3] and Kawakami^[4]. In distinction from [5], in these papers two neutrinos are already connected a priori in a "rigid" manner with the electron and muon, respectively (see below).

²⁾The statement (2) is essentially based on the assumption that in determining the form of the basic weak interaction all leptons can be considered massless (cf. [9]).

$$H_S \sim (\bar{\nu}_e (1 + \gamma_5) \mu^-) (\bar{e}^- (1 - \gamma_5) \tilde{\nu}_\mu) \quad (8)$$

are identical with each other. The tensor interaction

$$H_T \sim (\bar{\nu}_e \gamma_\alpha \gamma_\beta (1 + \gamma_5) \mu^-) (\bar{e}^- \gamma_\alpha \gamma_\beta (1 - \gamma_5) \tilde{\nu}_\mu), \quad \alpha \neq \beta \quad (9)$$

vanishes as a consequence of Eq. (2). This means that within the framework of leptonic interactions only (the muon decay theory) one is allowed to consider with equal rights both the vector and scalar currents.

The situation is different for the interaction of leptons with baryons. It is easy to see that in this case giving up the assumption of V-coupling would lead to a loss of uniqueness of the lepton pairings (4). In other words, if one does not impose V-coupling, the conservation laws of the charges and γ_5 -invariance would allow not only the pairings (4) but also the following lepton pairings:

$$L_3 = (\tilde{\nu}_\mu, e^-), \quad L_4 = (\tilde{\nu}_e, \mu^-). \quad (10)$$

Thus, for instance, the beta-decay Hamiltonian could be of the form

$$H_\beta \sim (\bar{p} (1 \pm \gamma_5) n) (\bar{e}^- (1 - \gamma_5) \tilde{\nu}_\mu). \quad (11)$$

However, this possibility contradicts the experiments on angular distribution and the electron asymmetry in beta decay. Thus, in distinction from the theory of muon decay, a complete theory of the strangeness-conserving weak interactions of leptons and baryons is uniquely determined (and in agreement with experiment) only if one adds the independent assumption of V-A-coupling to the conservation of electric, baryonic and leptonic charge and γ_5 -invariance.

The conclusion that in the leptonic interactions of strange particles with $\Delta S = +\Delta Q$ the coupling is also of the V-A type follows less definitely from experiment^[8]. However, the possibility is not excluded that the (V-A) coupling is not valid for the recently discovered^[12] strange particle decays with a strangeness change $\Delta S = -\Delta Q$. This would not contradict a limited universality, since, according to^[13] a combination of $\Delta S = +\Delta Q$ currents is impossible (the absence of transitions with $\Delta S = 2$).

Marshak and Sudarshan^[8] were the first to note that the (V-A) coupling in the four fermion interaction of any four particles^[8,14] is a consequence of the requirement of invariance with respect to independent γ_5 -transformations for each particle. Such an assertion is however ambiguous, since it is not clear what is meant by "particle," or in other words to what to connect the sign of chirality, to the electric charge, as in Eq. (2), or

to the leptonic (baryonic) charge. This ambiguity is removed in part in the γ_5 -classification of lepton doublets proposed in^[9], which generalizes the lepton classification (1)–(3). It follows from this classification that in the absence of renormalizations, the $\Delta S = -\Delta Q$ baryon currents can be only scalar or tensorial and cannot be vectorial (owing to the different chiralities of the nucleon and of the Σ^+ hyperon). It is a characteristic feature that the coupling of these currents to the leptonic currents can only be scalar, since the tensor coupling vanishes.

In the present paper we consider the consequences of the assumption that in leptonic interactions the $\Delta S = +\Delta Q$ and $\Delta S = -\Delta Q$ currents of the strongly interacting particles, are vectorial and scalar (or tensorial) respectively. In this case the $\Delta S = -\Delta Q$ leptonic decays of strange particles should contain the leptonic pairings (10), i.e., the electron should appear together with the muonic antineutrino $\tilde{\nu}_\mu$ and the muon with the electronic antineutrino $\tilde{\nu}_e$, e.g.,

$$K^0 \rightarrow \pi^+ + e^- + \nu_\mu. \quad (12)$$

A decisive test of this assumption could be obtained in high energy neutrino experiments in the reactions

$$\tilde{\nu}_\mu + n \rightarrow e^- + \Sigma^+, \quad \nu_\mu + p \rightarrow n + K^0 + e^+, \quad (13)$$

$$\tilde{\nu}_e + n \rightarrow \mu^- + \Sigma^+, \quad \nu_e + p \rightarrow n + K^0 + \mu^+. \quad (14)$$

At present the reactions (13) could possibly be observed experimentally, since their threshold differs only slightly from the threshold of the reaction which has already been observed in Brookhaven:

$$\tilde{\nu}_\mu + p \rightarrow n + \mu^+, \quad (15)$$

and their cross sections are possibly by one or two orders of magnitude smaller.

In addition to the neutrino experiments at high energies, an experimental test of the hypothesis could be obtained from an investigation of the decays of $K_{1,2}^0$ mesons:

$$K_1^0 \rightarrow \pi^\pm + e^\mp + \nu, \quad (16)$$

$$K_2^0 \rightarrow \pi^\pm + e^\mp + \nu. \quad (17)$$

Owing to the absence of interference between the decays of the K^0 and $\tilde{K}^0$ components of $K_{1,2}^0$ the total probabilities of the K_1^0 - and K_2^0 -meson decays (16) and (17) must be equal. Apparently experimental data^[12] point to the absence of such an equality.

The presence of S- or T-coupling in the $\Delta S = -\Delta Q$ decay modes of strange particles should also show up in the K_{e3} and $K_{\mu 3}$ decays of type

(17) (cf. [16,17]). The existing experimental data^[16] speak in favor of V-coupling in these processes, but apparently do not allow to rule out an admixture of (S, T)-coupling of the order of 30–40%, which does not contradict the experimental data regarding the relative probabilities of the $\Delta S = +\Delta Q$ leptonic decays^[12]. An increase of the accuracy of measurements of the angular correlation of leptons in these experiments would be desirable, especially for angles $\theta > \pi/2$ [17]. In case a violation of the V-coupling exists in any weak interaction process (as well as in the case of a nonvanishing neutrino mass) transitions $\nu_e \longleftrightarrow \tilde{\nu}_\mu$ would be possible in higher orders of the weak interaction constant.

Should it be proved experimentally that the leptonic $\Delta S = -\Delta Q$ modes of the strange particles are, like the other interactions, of the vector type, then under the assumptions (1)–(3) made above, this would imply that the lepton pairings (4) are the only ones possible in all weak processes. In this case the predictions of the scheme of weak interactions with one leptonic charge will coincide with the predictions of the scheme with two leptonic charges (cf. [18]).

If the weak interactions proceed via intermediate bosons, there should exist, besides the vector bosons, also scalar bosons coupled to the leptonic currents and to the $\Delta S = -\Delta Q$ currents of the strongly interacting particles, e.g.,

$$H_s = \frac{1}{2}g_V [(\bar{n}(1 - \gamma_5)\Sigma^+) + (\bar{e}^-(1 - \gamma_5)\tilde{\nu}_\mu) + (\bar{\mu}^-(1 - \gamma_5)\tilde{\nu}_e)] B^- + \text{c.c.} \quad (18)$$

It has been assumed here for estimating purposes that the coupling constant of the scalar bosons to the baryon and lepton currents is one half of the coupling constant g_V between the vector bosons and the conserved vector current of the strongly interacting particles. At the same time it is necessary to consider that the coupling constant of the vector bosons to the leptonic currents is also equal to $g_V/2$, in order to maintain the equality of the effective coupling constants in beta decay and in muon decay³⁾.

If the weak interactions are mediated by bosons, then the muonic neutrinos can also generate electrons in collisions with nuclei in association with scalar bosons, in the following processes:

³⁾Such a choice of the coupling constants of the lepton currents with the vector and scalar bosons, which satisfies the condition of equality of the effective coupling constants in beta decay and muon decay, is symmetric but is not the only one possible.

$$\nu_\mu + Z \rightarrow Z + e^+ + B^-, \quad \tilde{\nu}_\mu + Z \rightarrow Z + e^- + B^+. \quad (19)$$

The generation of muons in association with vector bosons will be a competing process.

The assumption that in the strangeness-changing processes the electron could be associated with a muonic neutrino and the muon with an electronic neutrino has been made in the papers by Bludman^[19] and Feinberg, Gürsey, and Pais^[15]. The present scheme differs from the ones mentioned in the following essential conclusions.

1. The only possible lepton pairings are of the forms (4) and (10), but not (ν_μ, e^-) and (ν_e, μ^-) as considered in [19,15].

2. The “anomalous” pairs (10) interact with the $\Delta S = -\Delta Q$ currents but not with the $\Delta S = +\Delta Q$ currents.

3. The leptonic currents which contain the pairs (10) are not vector currents but scalar or tensor currents. The same is true for the $\Delta S = -\Delta Q$ baryonic currents.

4. In [18,15] two types of muon decay are possible: $\mu^- \rightarrow e^- + \tilde{\nu}_e + \nu_\mu$ and $\mu^- \rightarrow e^- + \nu_e + \tilde{\nu}_\mu$ and in the present case only one is possible, namely the first. Due to this fact in those schemes the process $\mu \rightarrow e + \gamma$ is possible in higher orders whereas in the present scheme this process is rigorously forbidden.

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Note added in proof (March 14, 1963). Should it be proved experimentally that the $\Delta S = -\Delta Q$ baryon currents are indeed scalar or tensorial in character and that the total probabilities of the decays (16) and (17) are not equal (the existence of interference between K^0 and $\bar{K}^0$ decays), this would mean that if V-coupling is violated the chiralities of the electron and muon change signs, and therefore, in principle, the weak interactions could be a source of the bare mass of these particles.

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THE PROBLEM OF PAIRINGS WITH NONZERO ANGULAR MOMENTUM IN A FERMI SYSTEM

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The properties of a Fermi system with an interaction in the case of pairs of nonzero angular momentum are investigated. A decoupling of the three-particle Green's function is proposed which allows one to obtain solutions both with an isotropic and an anisotropic gap. It is shown that the solution with an anisotropic gap is asymptotically exact. A discussion is given of the difficulties arising in obtaining a basis for solutions with an isotropic gap.

1. INTRODUCTION

THE problem of pairings with nonzero angular momentum in a Fermi system has been repeatedly discussed recently^[1-10] in connection with the problem of the superfluidity of He³. The author has shown^[11] that production of Cooper pairs in the triplet state with a total component of spin equal to zero is possible in antiferromagnetic substances as a result of the interaction between conduction electrons induced by spin waves. Possibly such a mechanism is responsible for the occurrence of superconductivity in ruthenium and osmium, for which no isotopic effect has been found^[12,13].

The theory of Fermi systems with an interaction in the case of pairs of nonzero angular momentum is developed on the basis of the reduced BCS Hamiltonian^[14]:

$$H_{red} = \sum_{p,\sigma} \xi_p a_{p\sigma}^+ a_{p\sigma} + \frac{1}{2V} \sum_{p,p',\sigma} V(p, p') a_{p'\sigma}^+ a_{p-\sigma}^+ a_{p\sigma} a_{p'-\sigma} \quad (1)$$

Here ξ_p is the kinetic energy of a fermion of momentum p referred to the Fermi energy, V is the volume of the system, $a_{p\sigma}$, $a_{p\sigma}^+$ are the creation and annihilation operators for a fermion of momentum p and spin component σ . The "potential" of the interaction $V(p, p')$ in the isotropic model depends only on the scattering angle: $V(p, p') = V(\mathbf{n}\mathbf{n}')$, where $\mathbf{n} = \mathbf{p}/p$, and differs from zero in an energy shell of thickness 2ω near the Fermi surface.

For the investigation of the ground state of the system with the "reduced" interaction Gor'kov and Galitskiĭ^[3] have generalized the technique of decoupling the chain of equations for the Green's function developed by Gor'kov^[15] to the case of conden-

sation of Cooper pairs with nonzero relative angular momentum. In this paper we propose a solution with an isotropic gap. Other authors^[1,2,6-10] assume that in the ground state the gap is anisotropic in spite of the fact that the original Hamiltonian is isotropic. In particular, Anderson and Morel have obtained this result with the aid of a variational principle by minimizing the average value of the energy over states of the Bardeen type:

$$\Phi = \prod_p (u_p + v_p a_p^+ a_{-p}^+) \Phi_0,$$

where Φ_0 is the "vacuum" amplitude.

In the next section we shall utilize a technique of calculation which differs inappreciably from the technique of Gor'kov and Galitskiĭ^[3], and we shall obtain, by decoupling the chain of equations for the Green's functions, solutions both with an isotropic and with an anisotropic gap. In Sec. 3 it is shown that the anisotropic solution is asymptotically exact for $V \rightarrow \infty$. We shall also describe the difficulties arising in attempting to find a basis for the isotropic solutions.

2. STATES WITH ISOTROPIC AND ANISOTROPIC GAP

The energy spectrum of the system is determined by poles of the Fourier transform of the Green's function

$$G(p, t - t') = -i \langle T a_{p\sigma}(t) a_{p\sigma}^+(t') \rangle \equiv \langle\langle a_{p\sigma}(t); a_{p\sigma}^+(t') \rangle\rangle, \quad (2)$$

where the angle brackets denote averaging over a Gibbs ensemble.

The equation of motion for the function $G(p, t - t')$ has the form

$$i \frac{d}{dt} G(\mathbf{p}, t-t') = \delta(t-t') + \zeta_p G(\mathbf{p}, t-t') - \frac{1}{V} \sum_{\mathbf{p}'} V(\mathbf{p}, \mathbf{p}') \Gamma(\mathbf{p}, \mathbf{p}', t-t'), \quad (3)$$

where Γ is the two-particle Green's function:

$$\Gamma(\mathbf{p}, \mathbf{p}', t-t') = \langle\langle a_{-\mathbf{p},-\sigma}^+(t) a_{-\mathbf{p}',-\sigma}(t) a_{\mathbf{p}\sigma}(t); a_{\mathbf{p}\sigma}^+(t') \rangle\rangle. \quad (4)$$

We set up the equation of motion for the function Γ :

$$i \frac{d}{dt} \Gamma(\mathbf{p}, \mathbf{p}', t-t') = -\zeta_p \Gamma(\mathbf{p}, \mathbf{p}', t-t') - \frac{1}{V} \sum_{\mathbf{p}''} V(\mathbf{p}, \mathbf{p}'') \langle\langle a_{\mathbf{p}\sigma} a_{\mathbf{p}''\sigma}^+ a_{-\mathbf{p}',-\sigma}^+ a_{-\mathbf{p}',-\sigma} a_{\mathbf{p}\sigma}; a_{\mathbf{p}\sigma}^+(t') \rangle\rangle + \langle\langle a_{-\mathbf{p},-\sigma}^+ i \frac{d}{dt} (a_{-\mathbf{p}',-\sigma} a_{\mathbf{p}\sigma}); a_{\mathbf{p}\sigma}^+(t') \rangle\rangle. \quad (5)$$

We shall achieve the simplest interpolation decoupling of three-particle Green's functions appearing in (5) by pairing the operators $a_{\mathbf{p}\sigma}$ and $a_{\mathbf{p}\sigma}^+$. In order to simplify the resulting system of equations we impose the additional condition

$$\langle\langle a_{-\mathbf{p},-\sigma}^+ i \frac{d}{dt} (a_{-\mathbf{p}',-\sigma} a_{\mathbf{p}\sigma}); a_{\mathbf{p}\sigma}^+(t') \rangle\rangle = 0. \quad (6)$$

The solutions which we shall obtain below satisfy this condition. Thus, we shall obtain a closed system of equations for $G(\mathbf{p}, t-t')$ and $\Gamma(\mathbf{p}, \mathbf{p}', t-t')$ which on going over to Fourier components assume the form

$$(\omega - \zeta_p) G(\mathbf{p}, \omega) + \frac{1}{(2\pi)^3} \int d^3\mathbf{p}' V(\mathbf{p}, \mathbf{p}') \Gamma(\mathbf{p}, \mathbf{p}', \omega) = 1, \quad (7)$$

$$(\omega + \zeta_p) \Gamma(\mathbf{p}, \mathbf{p}', \omega) = \frac{i}{(2\pi)^4} G(\mathbf{p}, \omega) \int d^3\mathbf{p}'' d\omega V(\mathbf{p}, \mathbf{p}'') \Gamma(\mathbf{p}'', \mathbf{p}', \omega). \quad (8)$$

A. States with isotropic gap. By taking into account the symmetry of the desired solution we shall assume $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ to be a function of ζ, ζ', ω and of the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$. We expand $V(\mathbf{p}, \mathbf{p}')$ and $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ in a series of spherical harmonics:

$$V(\mathbf{p}, \mathbf{p}') = \sum_l (2l+1) V_l P_l(\cos \theta) = 4\pi \sum_{l,m} V_l Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}'), \quad (9)$$

$$\Gamma(\mathbf{p}, \mathbf{p}', \omega) = \sum_l \Gamma_l(\zeta, \zeta', \omega) P_l(\cos \theta) = 4\pi \sum_{l,m} \frac{1}{2l+1} \Gamma_l(\zeta, \zeta', \omega) Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}'); \quad (10)$$

$P_l(\cos \theta)$ are the Legendre polynomials. On substituting these expressions into (7) and (8), we shall obtain equations for $G(\mathbf{p}, \omega)$ and for the coefficients in the expansion of the function $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$:

$$(\omega - \zeta) G(\mathbf{p}, \omega) + \sum_l \rho_l \int d\zeta' \Gamma_l(\zeta, \zeta', \omega) = 1, \quad (11)$$

$$(\omega + \zeta) \Gamma_l(\zeta, \zeta', \omega) = \frac{i}{2\pi} G(\mathbf{p}, \omega) \rho_l \int d\zeta'' d\omega \Gamma_l(\zeta'', \zeta', \omega). \quad (12)$$

Here we have introduced the notation $\rho_l = V l p_0^2 / 2\pi^2 v_0$, where p_0, v_0 are the Fermi momentum and velocity.

The function $G(\mathbf{p}, \omega)$ whose poles determine the spectrum of single particle excitations can be easily found:

$$G(\mathbf{p}, \omega) = \frac{\omega + \zeta_p}{\omega^2 - \epsilon_p^2} + i \frac{\pi}{2} \text{th} \frac{\beta \epsilon_p}{2} \left\{ \left(1 - \frac{\zeta_p}{\epsilon_p}\right) \delta(\omega + \epsilon_p) - \left(1 + \frac{\zeta_p}{\epsilon_p}\right) \delta(\omega - \epsilon_p) \right\}, \quad (13)^*$$

the imaginary part is determined in accordance with Landau's theorem^[16], $\epsilon_p = (\zeta_p^2 + \Delta^2)^{1/2}$ is the energy of an elementary excitation, β is the reciprocal temperature, and

$$\Delta^2 = -\frac{i}{2\pi} \sum_l \rho_l^2 \int \Gamma_l(\zeta, \zeta', \omega) d\zeta d\zeta' d\omega. \quad (14)$$

On dividing Eq. (12) by $\omega + \zeta$ and on integrating over ζ, ζ', ω , we obtain

$$A_l \left(1 - \rho_l \int_0^\omega d\zeta \frac{1}{\epsilon} \text{th} \frac{\beta \epsilon}{2}\right) = 0, \quad (15)$$

$$A_l = \int \Gamma_l(\zeta, \zeta', \omega) d\zeta d\zeta' d\omega. \quad (16)$$

The system (15) admits the trivial solution $A_l = 0$ for all l corresponding to the normal state. There also exists a number of nontrivial solutions determined by equations

$$1 = \rho_{l_0} \int_0^\omega d\zeta \frac{1}{\epsilon} \text{th} \frac{\beta \epsilon}{2}, \quad \rho_{l_0} > 0, \quad A_l = 0 \text{ for } l \neq l_0. \quad (17)$$

In particular, at the temperature $T = 0$ we obtain from these equations

$$\Delta(0) = 2\omega \exp(-1/\rho_{l_0}), \quad (18)$$

which coincides exactly with the solution of Gor'kov and Galitskiĭ^[3]. The maximum gap Δ , and consequently, the thermodynamically most advantageous state corresponds to maximum ρ_l . The ratio of the critical temperature T_C to $\Delta(0)$ is the same as in the case of a pairing with $l = 0$: $T_C = 1.75 \times \Delta(0)$.

So far we have not used the additional condition (6) which can be written in the form

$$2\epsilon' \Gamma_l(\zeta, \zeta', \omega) = \rho_l \text{th} \frac{\beta \epsilon'}{2} \int d\zeta'' \Gamma_l(\zeta, \zeta'', \omega). \quad (19)$$

* $\text{th} = \tanh$.

It can be easily shown that equations (11), (12), (19), can be satisfied simultaneously¹⁾ by setting all the Γ_l except Γ_{l_0} equal to zero, and

$$\begin{aligned}\Gamma_{l_0}(\zeta, \zeta', \omega) &= \frac{i}{2\pi} F_{l_0}(\zeta, \omega) \int F_{l_0}^*(\zeta', \omega) d\omega \\ &= i F_{l_0}(\zeta, \omega) F_{l_0}^*(\zeta'),\end{aligned}\quad (20)$$

where

$$F_{l_0}(\zeta, \omega) = \frac{i\Delta}{\omega + \zeta} G(p, \omega). \quad (21)$$

A physical interpretation of the solution obtained was proposed by Gor'kov and Galitskii^[3]. The decoupling of the function $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ adopted by us means that the operator $\psi^+ \psi^+$ can create a pair of relative angular momentum l , while the component of the angular momentum m can be arbitrary. Averages of four Fermi operators can be represented in the present scheme in the following manner:

$$\begin{aligned}\langle \psi \psi \psi^+ \psi^+ \rangle &= \sum_m \langle N | \psi \psi | N+2, l, m \rangle \langle N+2, l, m | \psi^+ \psi^+ | N \rangle, \\ &\quad (22)\end{aligned}$$

where the state $|N+2, l, m\rangle$ is obtained from the state $|N\rangle$ of a system of N particles by the addition of a pair of relative angular momentum l and component m .

B. States with anisotropic gap. Anisotropic solutions can be obtained if we assume that the function $\Gamma(\mathbf{p}, \mathbf{p}', \omega)$ can be factorized:

$$\Gamma(\mathbf{p}, \mathbf{p}', \omega) = i F(\mathbf{p}, \omega) F^*(\mathbf{p}'), \quad (23)$$

where

$$F(\mathbf{p}) = \frac{1}{2\pi} \int d\omega F(\mathbf{p}, \omega). \quad (24)$$

From relation (8) one can easily show that

$$(\omega + \zeta_p) F(\mathbf{p}, \omega) = \frac{i}{(2\pi)^4} G(\mathbf{p}, \omega) \int d^3\mathbf{p}' d\omega V(\mathbf{p}, \mathbf{p}') F(\mathbf{p}', \omega). \quad (25)$$

The function $G(\mathbf{p}, \omega)$ is of the form (13) where, however, $\epsilon_{\mathbf{p}} = (\zeta_{\mathbf{p}}^2 + |\Delta(\mathbf{p})|^2)^{1/2}$, while the function $\Delta(\mathbf{p})$ can be expressed in terms of $F(\mathbf{p}, \omega)$ in the following manner:

$$\Delta(\mathbf{p}) = \frac{1}{(2\pi)^4} \int d^3\mathbf{p}' d\omega V(\mathbf{p}, \mathbf{p}') F(\mathbf{p}', \omega). \quad (26)$$

By eliminating the function $F(\mathbf{p}, \omega)$ we obtain the equation for the energy gap $\Delta(\mathbf{p})$:

¹⁾Generally speaking, there is no need to give such a proof, since the system of equations for a pairing of angular momentum $l = l_0$ is formally the same as in the case $l = 0$, when the solution is well known to exist.

$$\Delta(\mathbf{p}) = \frac{\pi}{(2\pi)^4} \int d^3\mathbf{p}' V(\mathbf{p}, \mathbf{p}') \frac{\Delta(\mathbf{p}')}{\epsilon(\mathbf{p}')} \operatorname{th} \frac{\beta \epsilon_{\mathbf{p}'}}{2}, \quad (27)$$

which, generally speaking, is complex [a physical meaning can be ascribed to the absolute value of the function $\Delta(\mathbf{p})$]. The energy gap $\Delta(\mathbf{p})$ depends only on the direction of the vector $\mathbf{p}$: $\Delta(\mathbf{p}) = \Delta(\mathbf{n})$. On expanding $\Delta(\mathbf{n})$ in terms of spherical harmonics:

$$\Delta(\mathbf{n}) = \sum_{l, m} \Delta_{lm} Y_{lm}(\mathbf{n}), \quad (28)$$

we shall obtain for the expansion coefficients Δ_{lm} a system of transcendental equations:

$$\begin{aligned}\Delta_{lm} &= \rho_l \sum_{l', m'} \Delta_{l'm'} \int_0^\omega d\zeta' \int d\mathbf{n}' \\ &\times \frac{Y_{l'm'}(\mathbf{n}') Y_{lm}^*(\mathbf{n}')}{V \zeta'^2 + |\Delta(\mathbf{n}')|^2} \operatorname{th} \frac{\beta V \zeta'^2 + |\Delta(\mathbf{n}')|^2}{2}.\end{aligned}\quad (29)$$

The additional condition (6) can be rewritten in the following manner:

$$\begin{aligned}2\zeta' \Gamma(\mathbf{p}, \mathbf{p}', \omega) + \frac{1}{(2\pi)^3} \int d^3\mathbf{p}'' V(\mathbf{p}', \mathbf{p}'') \\ \times \{n(\mathbf{p}') \Gamma(\mathbf{p}, -\mathbf{p}'', \omega) \\ - [1 - n(-\mathbf{p}')] \Gamma(\mathbf{p}, \mathbf{p}'', \omega)\} = 0,\end{aligned}\quad (30)$$

where $n(\mathbf{p}) = \langle a_{\mathbf{p}\sigma}^+ a_{\mathbf{p}\sigma} \rangle$.

The anisotropic solution obtained by us satisfies Eq. (30) only if the function $F(\mathbf{p}, \omega)$, and consequently also $\Delta(\mathbf{p})$, is even (or odd)²⁾ with

$$F(\mathbf{p}, \omega) = \pm F(-\mathbf{p}, \omega) = \frac{i\Delta(\mathbf{p})}{\omega + \zeta_p} G(\mathbf{p}, \omega). \quad (31)$$

This means that among the solutions of the equation for $\Delta(\mathbf{p})$ only those are acceptable in which $\Delta(\mathbf{p})$ is expanded in terms of spherical harmonics with only even values of l (singlet state) or only with odd values of l (triplet state).

It is of interest to note that the critical temperature T_c , determined from the condition that the function $\Delta(\mathbf{p})$ vanishes identically, satisfies the same equation both in the case when the gap is isotropic and in the case when the gap is anisotropic. This equation is of the form

$$1 = \rho_{l_0} \int_0^\omega \frac{d\zeta}{\zeta} \operatorname{th} \frac{\zeta}{2T_c}, \quad (32)$$

where ρ_{l_0} is the maximum positive value of ρ_l . The critical temperature is equal to

$$T_c = 3.5 \omega \exp(-1/\rho_{l_0}). \quad (33)$$

Thus, if in the interaction $V(\mathbf{p}, \mathbf{p}')$ at least one

²⁾Incidentally, such symmetry properties of the function $\Delta(\mathbf{p})$ are obvious.

of the quantities V_l is positive, then at sufficiently low temperatures the system will exhibit the property of superfluidity (or superconductivity). A positive sign of V_l denotes attraction of a pair of fermions in a state of angular momentum l .

3. ASYMPTOTIC EXACTNESS OF SOLUTIONS IN THE CASE OF THE REDUCED HAMILTONIAN

The problem of the symmetry properties of the state of thermodynamic equilibrium of a system with interaction in pairs of angular momentum different from zero is at the present moment controversial^[1-3,6-10]. The solution with an isotropic gap proposed by Gor'kov and Galitskiĭ^[3] cannot be obtained, for example, by the variational method, since the structure of the wave function of the isotropic ground state, if the latter exists, is very complicated. In the paper by Hone^[7] it is shown that it is very difficult to construct a chain of many-particle isotropic Green's functions satisfying the equations of motion obtained with the aid of the reduced Hamiltonian. However, we note that such objections are not valid as proofs. At the same time it is asserted that if in the reduced Hamiltonian one neglects the kinetic energy, then the method of Anderson and Morel gives the correct value for the energy of the ground state^[8]. We shall show in this section that solutions with an anisotropic gap are asymptotically exact, and we shall describe the difficulties arising in efforts of finding a basis for the isotropic solutions.³⁾

In the preceding section by decoupling the three-particle Green's function we obtained a closed system of equations for the one- and two-particle Green's functions. We now turn to the solution of an infinite chain of equations for the Green's functions.

Following Bogolyubov^[17,18] we shall consider the temperature Green's functions

$$G_{\mathfrak{M}}(\mathbf{p}, t - t') = \langle\langle a_{\mathbf{p}\sigma}(t) \mathfrak{M}(t); a_{\mathbf{p}',\sigma}^+(t') \rangle\rangle, \quad (34)$$

$$\Gamma_{\mathfrak{N}}(\mathbf{p}, \mathbf{p}', t - t') = \langle\langle a_{-\mathbf{p},-\sigma}^+(t) a_{-\mathbf{p}',-\sigma}(t) a_{\mathbf{p},\sigma}^+(t) \mathfrak{N}(t); a_{\mathbf{p}\sigma}^+(t') \rangle\rangle, \quad (35)$$

where $\mathfrak{M}, \mathfrak{N}$ are products of the form

$$\dots a_{\mathbf{q}\sigma_1}^+ a_{\mathbf{q}\sigma_1} \dots a_{\mathbf{g}\sigma_2}^+ a_{-\mathbf{g},-\sigma_2} a_{\mathbf{h}\sigma_3} a_{-\mathbf{h},-\sigma_3} \dots,$$

where all the operators appearing in $G_{\mathfrak{M}}$ and $\Gamma_{\mathfrak{N}}$ are different.

We construct equations of motion for these functions:

$$\begin{aligned} i \frac{d}{dt} G_{\mathfrak{M}}(\mathbf{p}, t - t') &= \delta(t - t') \langle\mathfrak{M}\rangle + \xi_{\mathbf{p}} G_{\mathfrak{M}}(\mathbf{p}, t - t') \\ &+ \frac{1}{V} \sum_{\mathbf{p}'} V(\mathbf{p}, \mathbf{p}') \langle\langle a_{-\mathbf{p},-\sigma}^+ a_{\mathbf{p}',\sigma} a_{-\mathbf{p}',-\sigma} \mathfrak{M}; a_{\mathbf{p}\sigma}^+ \rangle\rangle \\ &+ \langle\langle a_{\mathbf{p}\sigma} i \frac{d\mathfrak{M}}{dt}; a_{\mathbf{p}\sigma}^+(t') \rangle\rangle, \end{aligned} \quad (36)$$

$$\begin{aligned} i \frac{d}{dt} \Gamma_{\mathfrak{N}}(\mathbf{p}, \mathbf{p}', t - t') &= -\xi_{\mathbf{p}} \Gamma_{\mathfrak{N}}(\mathbf{p}, \mathbf{p}', t - t') \\ &- \frac{1}{V} \sum_{\mathbf{p}''} V(\mathbf{p}, \mathbf{p}'') \langle\langle a_{\mathbf{p}\sigma} a_{\mathbf{p}'',-\sigma}^+ a_{-\mathbf{p}'',-\sigma} a_{-\mathbf{p}',-\sigma} \mathfrak{N}; a_{\mathbf{p}\sigma}^+(t') \rangle\rangle \\ &+ \langle\langle a_{-\mathbf{p},-\sigma}^+ i \frac{d}{dt} (a_{-\mathbf{p}',-\sigma} a_{\mathbf{p}',\sigma}) \mathfrak{N}; a_{\mathbf{p}\sigma}^+(t') \rangle\rangle \\ &+ \langle\langle a_{-\mathbf{p},-\sigma}^+ a_{-\mathbf{p}',-\sigma} a_{\mathbf{p}',\sigma} i \frac{d\mathfrak{N}}{dt}; a_{\mathbf{p}\sigma}^+(t') \rangle\rangle. \end{aligned} \quad (37)$$

In Eqs. (36) and (37) we can, preserving an accuracy up to terms of order $1/N$ (N is the number of particles in the system, $N/V = \text{const}$), neglect terms in which the same operator is repeated several times. Then the chain of equations (36) and (37) will be closed even though it is infinite.

We shall attempt to obtain an asymptotically exact solution of our chain of equations in the form

$$G_{\mathfrak{M}}(\mathbf{p}, t - t') = \langle\mathfrak{M}\rangle G(\mathbf{p}, t - t'), \quad (38)$$

$$\Gamma_{\mathfrak{N}}(\mathbf{p}, \mathbf{p}', t - t') = \langle\mathfrak{N}\rangle \Gamma(\mathbf{p}, \mathbf{p}', t - t'). \quad (39)$$

Since the averages

$$\langle id\mathfrak{M}/dt \rangle = \langle id\mathfrak{N}/dt \rangle = 0,$$

the chain of equations (36) and (37) reduces to the pair of equations for $G(\mathbf{p}, t - t')$ and $\Gamma(\mathbf{p}, \mathbf{p}', t - t')$ which on going over to Fourier components assume the form (7) and (8) [taking into account the additional conditions (6)]. Thus, the solutions obtained in the preceding section satisfy the chain of asymptotically exact equations for the Green's function.

The decoupling of (38) and (39) is justified in the case of the solution with an anisotropic gap since in this case all the many-particle Green's functions under consideration can be factorized and, in particular, $\Gamma(\mathbf{p}, \mathbf{p}', \omega) = iF(\mathbf{p}, \omega) F^*(\mathbf{p}')$. This shows that the properties of a superconducting system with an anisotropic gap can be described by means of a quadratic approximating Hamiltonian of the type of Hamiltonian proposed by Bogolyubov^[17,18]:

$$H = \sum_{\mathbf{p}, \sigma} \{ \xi_{\mathbf{p}} a_{\mathbf{p}\sigma}^+ a_{\mathbf{p}\sigma} + \sigma | \Delta(\mathbf{p}) | (a_{\mathbf{p}\sigma}^+ a_{-\mathbf{p},-\sigma}^+ + a_{-\mathbf{p},-\sigma} a_{\mathbf{p}\sigma}) \}, \quad (40)$$

which can be diagonalized with the aid of a canonical transformation.⁴⁾ The approximating Hamil-

³⁾The asymptotic exactness of the BCS solution was first proved by Bogolyubov.^[17,18] Recently the same result was also obtained by Klein^[19] and by Haag.^[20]

⁴⁾Solutions with an anisotropic gap can be obtained in the case of the Hamiltonian H_{red} with the aid of Bogolyubov's

tonian, however, is not suitable for obtaining solutions with an isotropic gap ($l \neq 0$), since averages of the type $\langle a_{\mathbf{p}\mathbf{a}} a_{\mathbf{p}\mathbf{a}}^{\dagger} a_{\mathbf{p}'\mathbf{a}}^{\dagger} a_{\mathbf{p}'\mathbf{a}} \rangle$ on being averaged using the Hamiltonian (40) would have the structure of products $F(\mathbf{p}) F^*(\mathbf{p}')$ which in the scheme of Gor'kov and Galitskii would denote pairing in an S-state.

However, if we seek such a system of Green's functions that

$$\Gamma(\mathbf{p}, \mathbf{p}', \omega) = \Gamma_l(\xi, \xi', \omega) P_l(nn'),$$

then the decoupling of (38) and (39) is unacceptable, since the symmetry of many-particle Green's functions is violated. For example, in the case of such decoupling averages of the type

$$\langle a_{\mathbf{p}_1}^{\dagger} a_{-\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_1} a_{-\mathbf{p}_1} \rangle, \langle a_{\mathbf{p}_1}^{\dagger} a_{-\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} a_{-\mathbf{p}_2}^{\dagger} a_{\mathbf{p}_2} a_{-\mathbf{p}_2} \rangle$$

turn out to be proportional to $P_l(n_1 n_1') P_l(n_2 n_2')$ and are altered by the interchange of arguments $n_1 \rightleftharpoons n_2$, i.e., the required symmetry is absent. Therefore, one should seek other methods of laying a basis for the isotropic solution.

In the preceding section we decoupled the three-particle Green's function. One can proceed somewhat further in this direction. If we restrict ourselves to a discussion of Green's functions in which products of the type $a_{\mathbf{g}\mathbf{a}}^{\dagger} a_{-\mathbf{g}\mathbf{a}}^{\dagger} a_{\mathbf{f}\mathbf{a}} a_{-\mathbf{f}\mathbf{a}}$ occur not more than twice, while the number of products of the type $a_{\mathbf{p}\mathbf{a}}^{\dagger} a_{-\mathbf{p}\mathbf{a}}^{\dagger}$ can be arbitrary, then one can pair the operators $a_{\mathbf{p}\mathbf{a}}^{\dagger}$ and $a_{-\mathbf{p}\mathbf{a}}$, by assuming that the four-particle Green's function is of the following form:

$$\begin{aligned} & \langle \langle a_{-\mathbf{p}}^{\dagger}, -a_{-\mathbf{p}'}^{\dagger}, -a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}'}^{\dagger}, -a_{\mathbf{p}'}^{\dagger} a_{-\mathbf{p}}^{\dagger}, a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}'}^{\dagger}, a_{\mathbf{p}'}^{\dagger} a_{-\mathbf{p}}^{\dagger}; a_{\mathbf{p}\mathbf{a}}^{\dagger}(t') \rangle \rangle \\ &= -i F_l(\xi, t - t') F_l^*(\xi') F_l(\xi_1) F_l^*(\xi'_1) [P_l(nn') P_l(n_1 n_1') \\ &+ P_l(nn'_1) P_l(n_1 n') - P_l(nn_1) P_l(n_1 n'_1)]. \end{aligned} \quad (41)$$

The functions $F_l(\xi, \omega)$ and $F_l(\xi)$ are defined by relations (20) and (21) of the preceding section. The expression (41) possesses the necessary symmetry.

In the third term of the right hand side of Eq. (36) as a result of integrating over $\mathbf{p}'$ using the weighting function $V(\mathbf{p}, \mathbf{p}')$ the dependence on $\mathbf{n}$ disappears in virtue of the relation

$$\int d\mathbf{n}' P_l(nn') P_l(n_1 n') = \frac{4\pi}{2l+1} P_l(nn_1), \quad (42)$$

canonical transformation. The present author^[11] has obtained the Hamiltonian for the interaction of conduction electrons of an antiferromagnetic substance with spin waves. By applying the method of canonical transformation to this Hamiltonian one can also obtain the energy spectrum for a system with an anisotropic gap.

and the integral turns out to be proportional to $\langle \mathcal{M} \rangle$, and this leads to the equation for $G(\mathbf{p}, \omega)$ discussed in the preceding section in the case of an isotropic gap. A similar situation holds also for Eq. (37).

Unfortunately, we cannot demonstrate a decoupling of the functions $G_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}}$ containing a greater number of operators of the type $a_{\mathbf{g}\mathbf{a}}^{\dagger} a_{-\mathbf{g}\mathbf{a}}^{\dagger} a_{\mathbf{g}'\mathbf{a}} a_{-\mathbf{g}'\mathbf{a}}$ and the question of whether there exist isotropic solutions of the infinite chain of equations remains open. It is possible that the scheme of Gor'kov and Galitskii is an approximate one even in the case of the reduced interaction.

Thus, at the present moment there have been proposed a number of solutions of the problem of pairings with nonzero angular momentum. For a given interaction $V(\mathbf{p}, \mathbf{p}')$ only one solution is acceptable. The problem of the choice of this solution is very complicated. The system (29) can be solved in the general case only by numerical methods. Vaks, Galitskii, and Larkin^[10] have succeeded in showing that certain particular anisotropic solutions are unstable. It is possible that an experimental investigation of superconducting ruthenium and osmium will clarify the situation.

I express my gratitude to L. P. Gor'kov for useful discussions.

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Letters to the Editor

QUASI-OPTICAL MODEL AND THE ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE

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It has been shown^[1] that the system of two particles in quantum field theory can be described by an equation of the Schrödinger type with a generalized complex potential. Such a quasi-optical description permits, on the one hand, the finding of the scattering matrix and, on the other, the study of the structure of bound and resonant states (virtual processes). If we pose the problem of describing only the relativistic scattering matrix, then the generalized potential will depend only on the energy and the momentum transfer of the system and the equation for the wave function will have the form

$$(E^2 - q^2 - m^2) \psi_{\pm}(q) = \frac{1}{V^{q^2 + m^2}} \int V^{\pm}(q, q'; E) \psi_{\pm}(q') d^3q', \quad (1)$$

where q and q' are the three-dimensional momenta, V^+ is the potential for states that are even in $\cos \theta$ (θ is the scattering angle), and V^- is the corresponding potential for odd states.

For simplicity we consider here the case of scalar particles. For the potential $V^{\pm}(q, q', E)$ we can take the following spectral representation:

$$V^{\pm}(q, q', E) = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{U^{\pm}(E, \nu)}{\nu + (q - q')^2} d\nu, \quad (2)$$

where $U(E, \nu)$ is a spectral function depending on the energy of the system and is complex in the region $E^2 > m_1^2$.

If the amplitude of the process $M(E, t)$ satisfies the dispersion relation in the time variable t for any fixed values of the energy E , then the projection of the amplitude $M^{\pm}$ on even and odd states will be represented in the form

$$M^{\pm}(E, t) = \int_{\mu^2}^{\infty} \frac{\sigma^{\pm}(E, \nu)}{\nu + (q - q')^2} d\nu. \quad (3)$$

With the aid of the method suggested in ^[1], we can thus construct a potential in the form (2). In ^[2] it was shown that for a description of bound states of the system we also can construct from field theory the potential of the form (2) without the hypothetical dispersion relations. In expression (3) we did not take into account the subtraction polynomial, but, depending on its degree n , our potential will correctly describe only waves with $l > n$, and, consequently, our conclusions on the asymptotic behavior hold if $n < 1$.

The imaginary part V characterizes the inelastic scattering processes. Regge showed that when the potential is a superposition of Yukawa potentials, the asymptotic behavior of the scattering amplitude, as $t \rightarrow \infty$, has the form

$$M(E, t) = g(E) t^{\alpha(E)}, \quad t = -(q - q')^2, \quad (4)$$

where q and q' are the initial and final momenta of the particle.

For investigation of the asymptotic behavior in the s channel, Eq. (1) must be written in the t channel: here t is connected with the energy through the relation $t = 4E^2$. Then the variable s will be the momentum transfer ($s \leq 0$).

In this note we would like to draw attention to the fact that the potential of type (2) leads to the Regge asymptotic behavior (4). This can be shown if we use the method developed by Fubini and Straffolini for ordinary potential scattering.^[3] We write the equation for the amplitude $T^{\pm}$:

$$T^{\pm}(q, q') = V^{\pm}((q - q')^2, E) + \int \frac{V^{\pm}((q - p)^2, E) T^{\pm}(p, q')}{[(E + ie)^2 - m^2 - p^2] \sqrt{p^2 + m^2}} d^3p. \quad (5)$$

(On the mass surface the transition amplitude is the same as in the ordinary scattering matrix.) We seek a solution of Eq. (5) in the form

$$T^{\pm}(q, q') = \frac{1}{\pi} \int_0^{\infty} \frac{\tau^{\pm}(q'^2, q^2, \nu)}{\nu - s} d\nu. \quad (6)$$

Substituting (2) and (6) into (5), we obtain an equation for the spectral function τ :

$$\begin{aligned} \tau^{\pm}(q'^2, q^2, \nu, E) &= U^{\pm}(E, \nu) \\ &+ \iint \frac{Q^{\pm}(q'^2, \nu, u', t', q^2, E)}{(E^2 - m^2 - u') \sqrt{u'^2 + m^2}} \tau^{\pm}(q'^2, u', t', E) du' dt'; \\ Q^{\pm}(q'^2, \nu, u', t', q^2, E) &= \frac{1}{2} \int K(q'^2, \nu, u', t', \nu_1, q^2) U^{\pm}(E, \nu_1) d\nu_1. \end{aligned} \quad (7)$$

Here K is a well-known kernel (see, for example, ^[4]).

In the asymptotic region $s \rightarrow \infty$, Eq. (7) takes the form

$$\begin{aligned} \tau_{as}^{\pm}(q'^2, q^2, \nu, E) &= \frac{1}{s} \int_0^s dv \int_0^{\infty} Q_{as}^{\pm}\left(\frac{\nu}{s}, q'^2, u', E\right) \frac{\tau_{as}^{\pm}(q'^2, u', \nu, E)}{(E^2 - m^2 - u') \sqrt{u' + m^2}} du', \\ Q_{as}^{\pm}(x, u, u', E) &= \int \frac{\theta(u' - ux - \nu_1 x / (1 - x)) U^{\pm}(E, \nu_1)}{(1 - x)^{1/2} (u' - ux - \nu_1 x / (1 - x))^{1/2}} d\nu_1. \end{aligned} \quad (8)$$

This equation has a solution of the form

$$\tau_{\alpha}^{\pm}(q'^2, q^2, \nu, E) = \tau_{\alpha}^{\pm}(q'^2, q^2, E) \nu^{\alpha(E)}. \quad (9)$$

Here the function τ_{α} satisfies the equation

$$\begin{aligned} \tau_{\alpha}^{\pm}(u, s, E) &= \int R_{\alpha}^{\pm}(u, u', s, E) \frac{\tau_{\alpha}^{\pm}(u', s, E)}{(E^2 - m^2 - u') \sqrt{u' + m^2}} du', \\ R_{\alpha}^{\pm}(u, u', s, E) &= \int U^{\pm}(E, \nu) dv \int_0^1 \frac{dx \cdot x^{\alpha}}{(1 - x)^{1/2}} \frac{\theta(u' - ux - \nu x / (1 - x))}{[u' - ux - \nu x / (1 - x)]^{1/2}}. \end{aligned} \quad (10)$$

From Eq. (10) we can determine the eigenfunction τ_{α} and the eigenvalue α , which is a function of E . For $E^2 < m_1^2$ the function $U(E, \nu)$ is real and, consequently, α is real.

Inserting (9) into (6) we obtain for large values of s

$$T(q'^2, q^2, s, E) = s^{z(E)} \tau_{\alpha}(q'^2, q^2, E) \frac{[1 + e^{-i\pi\alpha(E)}]}{\sin \pi\alpha(E)}. \quad (11)$$

We can also obtain similar results directly from Eq. (1) by going over to partial waves.^[3]

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NEUTRON-NEUTRON TOTAL CROSS SECTION AT 8.3 GeV

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THE neutron-neutron total cross section was measured at the proton synchrotron of the Joint Institute of Nuclear Research from the attenuation of a neutral beam under conditions of good geometry ($\theta/2 = 0.228^\circ$).

As a neutron detector we used a telescope consisting of scintillation counters and a Cerenkov counter with complete absorption in lead glass. The detector recorded only those neutrons which, in interactions in an aluminum converter 10 cm thick, produced secondary particles (mainly neutral and charged pions) whose energy release in the Cerenkov-counter radiator was somewhat greater than the threshold energy. The energy thresholds of the neutron counter were calibrated from measurements of the energy of accelerated protons in the accelerator and with an electron beam. A system of fast discriminators with a resolving time of 1.0 μ sec was used for the pulse-height analysis of the Cerenkov-counter output. A second identical system of discriminators permitted simultaneous counts of random coincidences. As a monitor we used a telescope consisting of three scintillation counters.^[1] The neutron-neutron cross section was measured by the difference method with 50.01- and 55.60-g/cm² targets of H₂O and D₂O.

To decrease the effect of fluctuations in the measuring equipment and in the accelerator during the measurements, the ordinary and heavy water targets were exposed alternately for approximately 10-12 cycles of accelerator operation.

The experimentally obtained value of the n-n total cross section at an effective energy of about 8.3 GeV is

$$\sigma_{nn} = 31.5 \pm 1.7 \text{ mb}$$

The error is statistical.

Glauber^[2] showed that the cross section for the interaction of high-energy particles with deuterons should be less than the sum of cross sections for free neutrons and protons. In order to obtain the true value for the total n-n cross section, it is necessary to take into account the effect of the

screening of the neutron by the proton in the deuteron. The correction for screening is very small for energies of the order of several hundred MeV when the de Broglie wavelength of the incident particles is large and when diffraction scattering does not lead to the formation of a shadow. For incident particle energies of the order of 1 GeV and above, the de Broglie wavelength of the particles is already much smaller than the interaction region, and the approximation of geometric optics is applicable here. Of course, the greater the inelastic cross section, the greater the screening effect. If the distance between nucleons in the deuteron is assumed to be much greater than the interaction region, we obtain the following expression for the total cross sections:

$$\sigma_{nd} = \sigma_{nn} + \sigma_{np} + 4\pi k^{-2} \operatorname{Re} [f_{nn}(0) f_{np}(0)] \langle r^{-2} \rangle, \quad (1)$$

where f_{mn} and f_{np} are the scattering amplitudes and $\langle r^{-2} \rangle$ is the mean value of the inverse of the square of the distance between the proton and the neutron in the deuteron. This expression can be simplified if it is assumed that the scattering amplitude at high energies is purely imaginary. Then $f(0) = ik\sigma/4\pi$, and substitution into (1) yields

$$\sigma_{nd} = \sigma_{nn} + \sigma_{np} - \frac{1}{4\pi} \sigma_{nn} \sigma_{np} \langle r^{-2} \rangle.$$

The quantity $\langle r^{-2} \rangle$ can be calculated theoretically if the deuteron wave function is known. Depending on the specific choice of this function, the values of the correction obtained by Glauber were -4.5 , -5.7 , and 7.2 mb.

The value of the correction for screening can also be estimated from the known experimental values of σ'_{nn} , σ'_{pp} , and σ'_{np} , σ'_{pn} measured at the same energy (σ' is the total nucleon-nucleon cross section without allowance for the correction due to screening in the deuteron). From the conditions of charge symmetry of the nuclear forces (if the electromagnetic interaction is neglected) the value of σ'_{pp} should be equal to σ'_{nn} (for free nucleons). It thus follows that $\sigma'_{pp} - \sigma'_{nn} = \sigma_{nn} - \sigma'_{nn}$.

In [3] it has been shown that, within the limits of error, $\sigma_t(pp)$ is constant in the 10–20 GeV/c energy interval and is equal to 39.5 ± 1 mb. Substitution of the experimental values of the total cross sections σ_{pp} and σ'_{nn} yields the correction due to screening: $\delta\sigma = 39.5 - 31.5 = 8$ mb, which is good agreement with the value 7.2 mb obtained by Glauber for a potential of the form $V(r) \sim \lambda(e^{\lambda r} - 1)^{-1}$.

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SUPERCONDUCTIVITY OF GALLIUM NITRIDE

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It is well known that among the nitrides of transition metals there are superconductors with reasonably high transition temperatures.^[1] Nitrides of non-transition elements have not yet been investigated. The present work deals with a study of the superconductivity of the nitrides of two non-transition metals, gallium and indium.

The nitrides were prepared by direct nitration of the metals and by reaction of the oxides with ammonia.^[2] In this way, we obtained both nitrides which were almost exactly stoichiometric, corresponding to the formulas GaN and InN, and compounds containing oxygen impurity.

X-ray diffraction analysis of the stoichiometric compounds showed that their lattices were hexagonal, of wurtzite type, with parameters in good agreement with the published data:

	a , Å	c , Å	c/a	
GaN:	3.182	5.173	1.626	} present work
InN:	3.540	5.706	1.611	
GaN:	3.180	5.166	1.622	[3]
InN:	3.540	5.705	1.611	

The superconductivity of GaN and InN was determined by a magnetic method using powder samples. To obtain the temperature dependence of the critical field, the magnetic moment of the samples was measured as a function of the magnetic field at several constant temperatures.

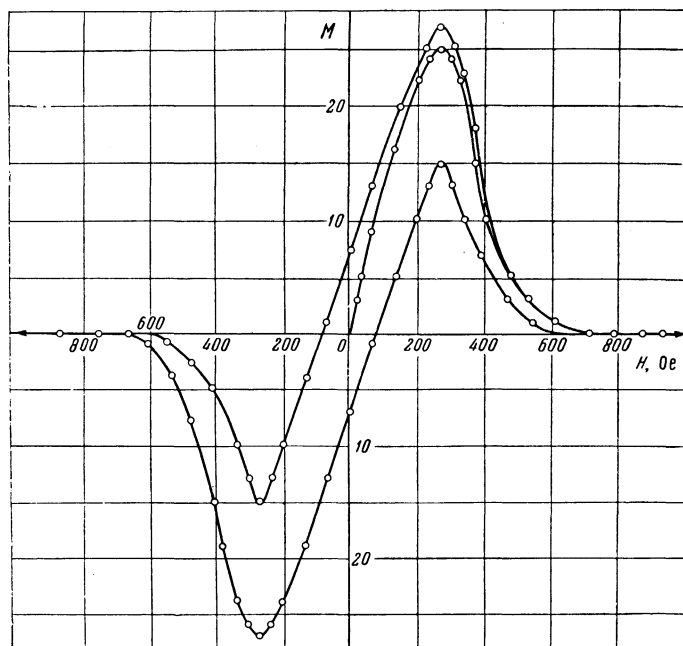


FIG. 1

Figure 1 gives the dependence of the magnetic moment (in arbitrary units) on the magnetic field for a GaN sample at $T = 3.2^\circ\text{K}$, and Fig. 2 gives the temperature dependence of the critical field for the same sample. If the dependence of the critical field (determined by extrapolation of the falling part of the magnetic moment curve to the temperature axis, and denoted by H_{C1}) is plotted as a function of the square of the temperature, the experimental points fit a straight line quite well. Figure 2 shows that the superconductivity of GaN is characterized by the following parameters:

$$T_c = 5.85^\circ\text{K}, \quad (H_{C1})_{T=0} \approx 725 \text{ Oe}, \quad dH_{C1}/dT \approx 190 \text{ Oe}/^\circ\text{K}.$$

If the value of H_{C2} is determined (H_{C2} is the field at which the magnetic moment of the sample vanishes) it is found that $dH_{C2}/dT \approx 450 \text{ Oe}/\text{deg K}$.

The parameters of other samples did not differ greatly from those listed above, except for the samples in which some of the nitrogen atoms were replaced by oxygen. Such replacement greatly reduces T_c , which becomes less than 1.38°K .

In contrast to the behavior of gallium nitride, no superconductivity could be found in any sample of indium nitride down to 1.38°K .

Comparison of the superconducting properties of transition-metal nitrides with those of GaN reveals the following features. Transition-metal nitrides have a broad transition to the superconducting state, their T_c values depend on the method of preparation, and their $M(H)$ dependence has a broad hysteresis loop. On the other hand, all the

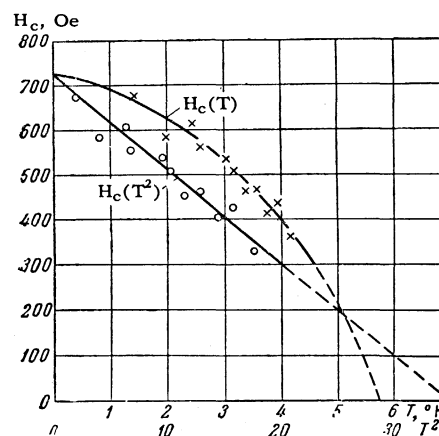


FIG. 2

samples of GaN, irrespective of their method of preparation, have the same value of T_c , and the dependence of their magnetic moment on the external field has a relatively narrow hysteresis loop, resembling the $M(H)$ dependence of pure metals. Gallium nitride is similar in its superconducting properties to compounds of bismuth with alkali metals (BiLi , BiNa , Bi_2K , Bi_2Rb , Bi_2Cs ^[5]) in which the superconductivity may be regarded as being due to an extremum in the dependence of T_c on the conduction electron density. According to the current theory this may be the same extremum as in the dependence of T_c on $N(0)$, where $N(0)$ is the density of states: $T_c = 1.14 \hbar \omega \exp \{ -1/N(0)V \}$,^[7] where V is the parameter representing the interaction of elec-

trons in the immediate vicinity of the Fermi surface.

The appearance of superconductivity in GaN can hardly be due only to the electron density extremum; in this case additional data should perhaps be used. Thus Samsonov and Neshpor^[8,9] noted a correlation between the superconducting transition points and the nature of the electron-density distribution in the crystal lattices of metal-like compounds of transition metals with non-metals such as boron, carbon, nitrogen, and silicon. This correlation appears as an increase of the transition temperatures of metal-like compounds having approximately the same crystal structure with increase of the asymmetry of the electron-density distribution in the lattice. This correlation is also supported by a reduction of the transition temperatures on increase of the relative content of the transition metal, i.e., in those cases where the role of the Me-Me bonds increases and the role of the Me-X bonds decreases (here Me is a metal and X is a metalloid).

Using these data for compounds of non-transition metals with non-metals we may postulate that these compounds can have high values of T_C due to the weaker Me-Me bonds compared with the transition metals. However, this is possible only when the proportion of covalent X-X bonds, which lower the conduction electron density, is not too high.

The superconductivity of the compound GaN is apparently the first case to be reported of superconductivity in nitrides of non-transition metals. As mentioned above, in contrast to transition-metal nitrides which have broad regions of homogeneity and consequently composition-dependent T_C , gallium nitride has no region of homogeneity and consequently its transition to superconductivity is sharp and there is one T_C for different samples of stoichiometric composition.

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THE GENERATION OF MILLIMETER WAVES IN RUBY BY OPTICAL PUMPING

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USUALLY the frequency of the auxiliary radiation for a quantum paramagnetic amplifier and generator (Q.P.A. and Q.P.G.) must be higher than the amplified or generated frequency. For making a Q.P.A. and Q.P.G. in the submillimeter range, the most realistic prospect in practice is to use auxiliary radiation in the optical range.

Recently, Devor et al.^[1] used the radiation from a pulsed ruby laser to excite a Q.P.A. that worked at a temperature of 4.2°K and a wavelength of 1.34 cm, with ruby as the working substance.

We have carried out experiments on the generation of millimeter wavelength radiation, in the range $(35-50) \times 10^9$ cps, by ruby cooled to liquid nitrogen temperature. A ruby laser, also operated at $T = 77^\circ\text{K}$, served as the pump. In Fig. 1b is shown the energy level scheme for a ruby crystal in an external magnetic field in the orientation we used—the c-axis perpendicular to H ($\theta = 90^\circ$). Figure 1a shows the energy level scheme for ruby in zero magnetic field (laser). The laser radiation component associated with the transition $\bar{E} \rightarrow {}^4A_2 (\pm \frac{1}{2})$ induces transitions of ions from level 3 to levels 5 and 6 (Fig. 1c). It should be borne in mind that the line width of the optical transition $\bar{E} \rightarrow {}^4A_2$ at liquid nitrogen temperature is $0.3-1 \text{ cm}^{-1}$, depending on the quality of the crystals used, and accurate matching of the transition frequencies is not required of the pump.

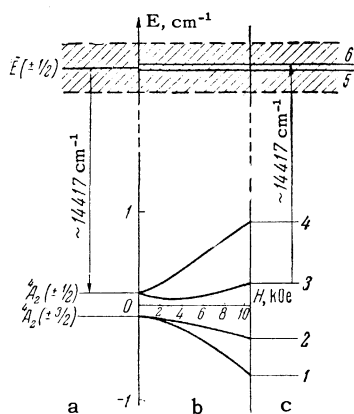


FIG. 1. The energy level scheme of the Cr^{3+} ion in ruby: a — zero magnetic field; b — magnetic field increases from left to right; c — energy levels for a magnetic field strength of 10,500 Oe. The shaded band corresponds approximately to the width of optical transition lines.

The experimental arrangement is schematically shown in Fig. 2. The radiation from a ruby crystal immersed in liquid nitrogen passed through a system of mirrors and a lens, and fell on another ruby crystal situated in a cutoff waveguide served as a resonator in the millimeter region. The crystal was cooled by liquid nitrogen through a thermal conductor. The laser radiation fell on the crystal through the open end of the cutoff waveguide. The high-frequency section of the apparatus consisted of a superheterodyne radio-spectrometer of the reflection type, with the aid of which the required value of magnetic field was set and the Q.P.G. radiation recorded. The pulse of laser radiation and the pulse of generated microwave power were observed simultaneously on the screen of a pulse oscilloscope. In Fig. 3 is shown one of the oscilloscope traces obtained with a laser radiation energy of ~ 0.1 joule. The microwave radiation appears in "spikes," as is frequently observed for a paramagnetic generator with normal radio-frequency pumping. The frequency of the spikes increased when the energy of the laser pulse was increased. The power of the

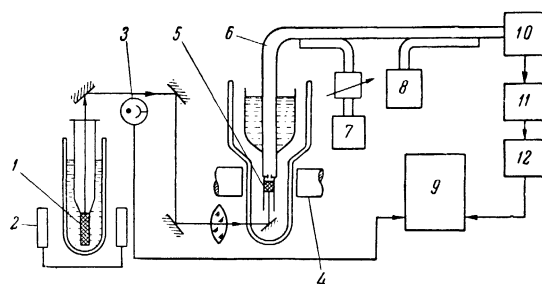


FIG. 2. Schematic diagram of the experimental arrangement: 1 — ruby crystal in laser; 2 — flash tube; 3 — photomultiplier; 4 — magnet; 5 — ruby crystal — Q.P.G. resonator; 6 — waveguide system; 7, 8 — klystrons; 9 — pulse oscilloscope; 10 — mixer; 11 — intermediate frequency amplifier; 12 — detector.

radiation in the millimeter range was $\sim 10^{-5}$ W.

The principal features of the operation of a Q.P.A. with optical pumping can be obtained from an analysis of the kinetic equations for a system of three energy levels, taking into account the large energy interval between the two lower levels (E_1 and E_2) and the third upper level.

Let N_1, N_2, N_3 be the populations of the three levels, W be the transition probability under the influence of the laser radiation field, w_{31} and w_{32} be the probabilities of spontaneous transitions into the lower level, w_{12} and w_{21} be the probabilities of relaxation transitions between the lower levels. Then

$$\begin{aligned} dN_1/dt &= (-N_1 + N_3)W - N_1w_{12} + N_2w_{21} + N_3w_{31}, \\ dN_2/dt &= N_1w_{12} - N_2w_{21} + N_3w_{32}, \\ dN_3/dt &= (N_1 - N_3)W - N_3(w_{31} + w_{32}). \end{aligned} \quad (1)$$

From these equations we can find the difference between the populations under steady state conditions. Putting

$$w_{32} = w_{31} = w_3, \quad h\nu/kT \ll 1, \quad w_{21} = w_{12}(1 + h\nu/kT),$$

we have

$$\Delta N = N_2 - N_1 = \frac{w_3/w_{12} - h\nu/kT - 2w_3h\nu/WkT}{3 + 4w_3/W + w_3/w_{12}} N, \quad (2)$$

where N is the total number of particles. By substituting in (2) the known probabilities of relaxation

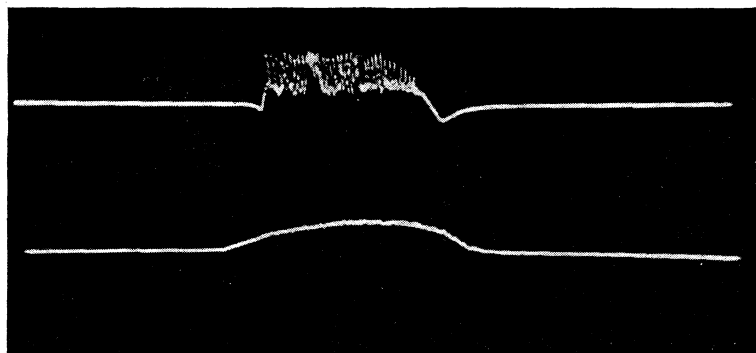


FIG. 3. Pulse of millimeter wave radiation (above), and of laser radiation (below). Sweep duration 500 μsec .

transitions in ruby at $T = 77^\circ\text{K}$ (the spin-lattice relaxation time $\tau_1 = 50 \mu\text{sec}$, the lifetime of the level $\bar{E}$ is $\sim 5 \text{ msec}$), we can show that it is impossible to obtain a steady-state negative temperature between the levels E_1 and E_2 . The negative temperature observed in our experiments, and the generation associated with it, are caused by nonstationary processes. Calculations show that generation cannot proceed for longer than $2-3 \tau_1$.

In the ideal case, if the spin-lattice relaxation process is neglected, it is possible to find the maximum energy that our three level system is able to radiate: it is $Nh\nu/6$. For our specimen with a chromium ion concentration of $\sim 0.05\%$, this energy is $\sim 2.5 \text{ erg}$. Since generation lasts $\sim 150 \mu\text{sec}$, the maximum radiation power should not exceed 1.7 mW . The generated power observed in the experiment indicates the large role of relaxation processes.

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COEXISTENCE OF ANTIFERROMAGNETIC AND SPECIAL DIELECTRIC PROPERTIES IN THE BiFeO_3 - LaFeO_3 SYSTEM

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THE compound BiFeO_3 is ferroelectric with a Curie temperature $T_C \sim 850^\circ\text{C}$ and antiferromagnetic with type G magnetic structure and Neel point $T_N \sim 380^\circ\text{C}$.^[1-4] The attention of a number of investigators^[5,6] has been turned recently toward a new class of substances combining ferroelectric and special magnetic properties and thereby having great scientific and practical interest. Complex investigations (x-ray, dielectric, and magnetic) of binary systems of compounds which to some degree or other combine special dielectric and magnetic properties are of great value for a thorough study of these phenomena.

The present paper describes results of such an investigation of the system BiFeO_3 - LaFeO_3 , in

which BiFeO_3 is a ferroelectric antiferromagnet, and LaFeO_3 is antiferromagnetic with weak ferromagnetism.^[7] The usual ceramic techniques were used to synthesize the samples. X-ray analysis of the samples at room temperature showed that a complete series of solid solutions with the perovskite structure is formed over the entire concentration range. These solid solutions exist in four modifications, depending on the LaFeO_3 content (in mole percent): up to 18.8—rhombohedral; from 18.8 to 55—pseudomonoclinic I (PMI); from 55 to 73—pseudomonoclinic II (PMII); above 73—pseudomonoclinic III (PMIII).

The magnetic measurements were carried out by the Faraday method from room temperature to 500°C in fields up to 8 kOe . The samples of all compositions demonstrated unusual magnetic properties.

Figure 1 shows the temperature dependence of the magnetization $\sigma(t)$ for different samples of the system in a field of 7600 Oe . Similar results for the magnetic susceptibility of BiFeO_3 as a function of temperature have been obtained by Smolenskiĭ et al.^[8]

In Fig. 2, besides the lines of the phase transition in this system (of which more will be said below), is presented the dependence of the room-temperature spontaneous magnetization σ_0 on the composition of the samples.

The presence of spontaneous magnetization in the compositions with 12.5, 15, and 17.5% LaFeO_3 , as well as in all investigated samples in the three pseudomonoclinic regions and the similarity between the magnetization vs. temperature curves for these compositions and the like curves for the weak ferromagnet LaFeO_3 , is evidence that all these samples are also antiferromagnets with weak ferromagnetism.

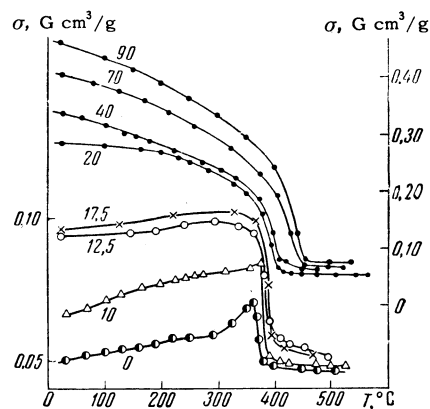


FIG. 1. Temperature dependence of the specific magnetization of samples of the system BiFeO_3 - LaFeO_3 in a field of 7600 Oe . The numbers on the curves indicate the LaFeO_3 content in mole %. The scale on the right is for compositions containing 20, 40, 70, and 90 mole % LaFeO_3 .

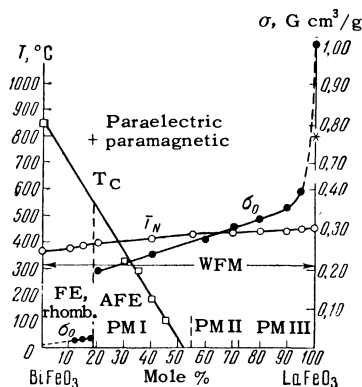


FIG. 2. Phase diagram of the system BiFeO_3 - LaFeO_3 and the dependence of the spontaneous magnetization at 20°C on the composition of the samples (FE – ferroelectric region, AFE – antiferro-electric, WFM – weakly ferromagnetic); * – approximate value of the spontaneous magnetization of LaFeO_3 at 20°C from the data of [9].

The spontaneous magnetizations of the sample of 10% LaFeO_3 and of pure BiFeO_3 were found to be commensurate within the limits of experimental error— $(8-9) \times 10^{-3} \text{ G cm}^3/\text{g}$. However, because of the presence of a sharp maximum in the temperature dependence of the magnetization for these compositions and the existence of spontaneous magnetization in samples of the rhombohedral modification, to which belong also both the 10% LaFeO_3 and pure BiFeO_3 , we conclude that these samples also are weakly ferromagnetic.

At the boundary between the rhombohedral and PMI modifications there is a step-like increase in the spontaneous magnetization. With further increase in the LaFeO_3 content up to 90% σ_0 , to a first approximation, increases linearly and continuously. A detailed study of the behavior of the spontaneous magnetization at the boundaries between PMI and PMII and between PMII and PMIII was not made, since our main interest was in the region of concentrations up to $\sim 50\%$ LaFeO_3 .

A sharper increase in the magnitude of the spontaneous magnetization is found to occur in the region of concentrations from 90 to 100% LaFeO_3 .

In addition to the x-ray and magnetic measurements, investigations of the temperature dependence of the dielectric constant were carried out on a number of samples whose conductivity was not too high for the measurements to be made.

On the basis of the totality of the results of all these measurements a phase diagram of the system was constructed (Fig. 2). The line T_C (Curie temperature) is the line for the phase transition from a ferro- or antiferroelectric state to a paraelectric one, and the line T_N (Néel point) is the line for the transition from a state with antiferromagnetic ordering to a paramagnetic state.

The following very interesting regions can be distinguished in the phase diagram: up to 18.8% LaFeO_3 the ferroelectric properties are combined with weak ferromagnetism, and in the region from 18.8 to 55% LaFeO_3 antiferroelectricity coexists with weak ferromagnetism, and the coexistence of electrical and magnetic ordering takes place in a wide interval of temperatures higher than room temperature. It should be emphasized that the sharp rise in spontaneous magnetization at the boundary between the rhombohedral and pseudomonoclinic I modifications with the transition from the ferroelectric to the antiferroelectric phase seems to us to be not accidental, but indicative of some interaction between the special dielectric and special magnetic states.

It is natural to expect coupling, since both characteristics depend strongly on the very same factor—the interatomic spacings. The existence of this coupling was theoretically predicted by Smolenskiĭ, [10] and the experimental results cited earlier [5,6] are evidence for it.

A more detailed exposition of the results of the x-ray, dielectric, and magnetic investigations and a discussion of them will be given in the journal “Kristallografiya” (Soviet Phys. Crystallography).

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