SATURATION EFFECTS FOR LONG-LIVED SYSTEMS IN SPATIALLY BOUNDED FIELDS

S. G. RAUTIAN and A. M. SHALAGIN

Semiconductor Physics Institute, Siberian Division, USSR Academy of Sciences; Novosibirsk State University

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Some regularities of emission by long-lived systems in spatially restricted fields are considered. It is demonstrated, in particular, that for weak saturation the width of the Lamb "dip" is determined by the natural line width Γ. The effect of recoil during photon emission and absorption on the structure of the "dip" is considered.

1. INTRODUCTION

IT is known that in the case of weak saturation in gas lasers, the Doppler-broadened line has a fine structure with width on the order of the natural line width Γ. It has been shown experimentally and theoretically[1-5] that in lasers using an absorbing gas cell, a plot of the power against the field frequency has a "peak" of width Γ (the Lisitsyn-Chebotaev "peak"). This effect uncovers great possibilities for increasing the frequency stability of laser radiation by decreasing the natural line width Γ.

Having in mind a frequency stability $\sim 10^{-14}$ or an absolute stability of ~ 1 Hz, it is natural to tend to a value $\Gamma \sim 10^3$ sec^{-1} and less (long-lived systems). In this case, however, most atoms traverse the transverse dimension d of the light beam within a time much shorter than $1/\Gamma$ (e.g., at d $\sim$ 1 cm, $\bar{v} \sim 10^5$ cm/sec we have $d/\bar{v} \sim 10^{-5}$ sec). Therefore atoms having a velocity u perpendicular to the light beam will give a structure whose width is determined by the larger of the two quantities, Γ or $u/d = 1/\tau$. The resultant width depends, naturally, on the distribution with respect to the velocity u and on the contribution made to the saturation by the atoms with velocity u.

Until now work on frequency stabilization in the radio and optical bands was oriented towards the use systems based on beams of atoms or molecules (e.g.,[6,7]). The width of the fine spectral structure then turned out to be $\sim 1/\bar{\tau}$, where $\bar{\tau}$ is the time of flight of the light beam at a mean-thermal velocity $\bar{v}$. To decrease the width, it was proposed in[7] to increase the effective $\bar{\tau}$, by causing the atom beam to pass through several light beams.

As is well known, the atom velocity distribution in beam systems is determined by the function $u^2 \exp(-u^2/\bar{v}^2)$, i.e., the number of slow atoms is very small. In the case of volume excitation, on the other hand, their number is much larger. Therefore the spectral structures in the latter case should be narrower than in beam systems. It is shown in the present paper that this narrowing is quite appreciable.

The question of the form of the spectral structures is closely connected with the manifestation of the recoil effect in the emission and absorption of a photon. It has been shown in[8] that the Lamb "dip" (or the Lisitsyn-Chebotaev "peak") splits into two components separated by a distance 2δ, where δ is the shift of the Bohr frequency upon emission or absorption of a photon:

$$\delta = \frac{k^2\hbar}{2m} = 1{,}25 \cdot 10^6 \lambda^{-2} M^{-1} \ [\text{sec}^{-1}]. \tag{1.1}$$

Here m and M are the mass and the atomic weight of the atom, k is the wave number, λ is the wavelength in microns. For $\lambda \sim 1\,\mu$ and for M on the order of several times ten, we have $\delta \sim 10^4$ sec^{-1}. In order for the recoil effect to be observed, it is necessary to have $\Gamma \lesssim 10^4$ sec^{-1}. Consequently, in the analysis of the recoil effect it is also necessary to take into account the transverse dimensions of the light beam.

Thus, the problem of emission of atoms in a spatially localized field is of interest from different points of view, and the present paper is devoted to an analysis of this question. We consider fields of such intensity that it is possible to confine oneself to the first order correction for saturation. It is also assumed that the atoms are excited with a Maxwellian velocity distribution, and the characteristic width of the distribution $\bar{v}$ satisfies the condition

$$\bar{v} \gg \Gamma / k, \tag{1.2}$$

which usually holds in gas lasers.

2. EVOLUTION OF TWO-LEVEL SYSTEM WHEN THE FIELD IS INSTANTANEOUSLY SWITCHED ON

For a better clarification of the meaning of the entire analysis that follows, let us consider the very simple problem of the "buildup" of an atomic oscillator after the field is instantaneously turned on. Let an ensemble of atoms, averaged over the excitation moments, interact with a traveling-wave field of the type $E = E_0 \exp\{i(\omega t - \mathbf{k}\cdot\mathbf{r})\}$. The field frequency ω is assumed to be close to the frequency ω_{mn} of the transition between the levels m and n. Assuming that the collisions do not change the velocity of the atom, we write the equations for the density matrix:

$$\begin{gathered}(\Gamma_j + d/dt)\rho_{jj} = q_j(v) \pm 2\mathrm{Re}\{iGe^{i(\Omega - \mathbf{kv})t}\rho_{mn}\}, \\ j = m, n, \quad \Omega = \omega - \omega_{mn}, \\ \left(\Gamma + \frac{d}{dt}\right)\rho_{mn} = iGe^{-i(\Omega - \mathbf{kv})t}(\rho_{mm} - \rho_{nn}), \quad G = \frac{d_{mn}E_0}{2\hbar},\end{gathered} \tag{2.1}$$

d_{mn} is the matrix element of the dipole moment; $q_j(v) = Q_j W_M(v)$; Q_j is the number of acts of excitation of the

level j per unit time and per unit volume[1]; $W_M(\mathbf{v})$ is the Maxwell distribution function with respect to the velocities v; Γ_j and Γ are the relaxation constants. We assume that the interaction with the field was turned on at the instant of time t = 0. Assuming the excitation $q_j(\mathbf{v})$ to be independent of the time, let us calculate by means of successive approximations (in terms of G^2) the work performed by the field per unit time:

$$W(t, \mathbf{v}) = -\hbar\omega 2\mathrm{Re}\{iG\rho_{mn}(t, \mathbf{v})\}. \tag{2.2}$$

In the approximation linear in G^2 we have

$$W^{(1)}(t, \mathbf{v}) = 2\hbar\omega G^2 N(\mathbf{v})\,\mathrm{Re}\left\{\frac{1 - e^{-[\Gamma + i(\Omega - \mathbf{kv})]t}}{\Gamma + i(\Omega - \mathbf{kv})}\right\}; \tag{2.3}$$

$$N(\mathbf{v}) = \left(\frac{Q_m}{\Gamma_m} - \frac{Q_n}{\Gamma_n}\right) W_M(\mathbf{v}),$$

where N(v) is the "unsaturated" stationary population difference. After a time $\sim 1/\Gamma$ expression (2.3) reaches a stationary value. The process of establishment of the stationary value can be regarded as a "buildup" of the oscillations of the atom, averaged over the moments of the excitation. Under condition (1.2), the mean value $\langle W^{(1)}(t, \mathbf{v})\rangle_v = W^{(1)}(t)$ reaches the stationary value

$$W^{(1)} = W^{(1)}(t = \infty) = 2\hbar\omega \frac{G^2 N_0 \sqrt{\pi}}{k\bar{v}} \exp\left\{-\left(\frac{\Omega}{k\bar{v}}\right)^2\right\} \tag{2.4}$$

$$N_0 = Q_m/\Gamma_m - Q_n/\Gamma_n,$$

within a characteristic time $1/k\bar{v} \ll 1/\Gamma$. This occurs because oscillations of different frequency are superimposed in the course of the averaging.

The correction for the saturation in expression (2.2), averaged over $\bar{v}$, has the form

$$W^{(2)}(t) = -\hbar\omega \frac{G^4 N_0 \sqrt{\pi}}{\Gamma k\bar{v}} \exp\left\{-\left(\frac{\Omega}{k\bar{v}}\right)^2\right\} \cdot \tag{2.5}$$

$$\times \left\{\frac{1}{\Gamma_m}\left[1 + \frac{\Gamma_m \exp\{-\Gamma t\} - \Gamma \exp\{-\Gamma_m t\}}{\Gamma - \Gamma_m}\right]\right.$$

$$\left. + \frac{1}{\Gamma_n}\left[1 + \frac{\Gamma_n \exp\{-\Gamma t\} - \Gamma \exp\{-\Gamma_n t\}}{\Gamma - \Gamma_n}\right]\right\}.$$

Unlike the preceding case, the characteristic "buildup" time (2.5) remains $\sim 1/\Gamma$ even after averaging with respect to v. This result is obvious, since the correction (2.5) is connected with the change of population, and the population cannot change more rapidly than within a time $\sim 1/\Gamma$. When $t < 1/\Gamma$ we obtain

$$W^{(2)}(t) = -\hbar\omega \frac{G^4 N_0 \sqrt{\pi}}{k\bar{v}} \exp\left\{-\left(\frac{\Omega}{k\bar{v}}\right)^2\right\} t^2. \tag{2.6}$$

The foregoing problem is physically equivalent to a stationary problem in which the atom has a specified velocity u along the axis x perpendicular to the wave vector k, and in addition G(x) is of the form

$$G(x) = \begin{cases} G & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2.7}$$

In this case it is necessary to make in the foregoing formulas the substitutions $t \to x/u$ and $d/dt \to ud/dx$. In particular, in place of (2.6) we get when $u > \Gamma x$

$$W^{(2)}(x) = -\hbar\omega \frac{G^4 N(u) \sqrt{\pi}}{k\bar{v}} \exp\left\{-\left(\frac{\Omega}{k\bar{v}}\right)^2\right\} \frac{x^2}{u^2}, \tag{2.8}$$

$$N(u) = N_0 W_M(u),$$

i.e., the contribution made to $W^{(2)}(x)$ by atoms with velocities $u > \Gamma x$ decreases sharply with increasing u. This fact is decisive in the problem of the structure of the Lamb "dip."

3. LOCALIZED FIELD. GENERAL EQUATIONS

Let us consider E(x, y, z; t) in the form

$$\mathbf{E}(x, y, z; t) = \mathbf{E}_0 g(x, y) \cos\omega t \cos kz, \tag{3.1}$$

where g(x, y) is a dimensionless function, equal to unity at the maximum. The equations for the density matrix elements are

$$[\Gamma + \partial/\partial t + (\mathbf{u} + \mathbf{v})\nabla]\rho_{mn} =$$
$$= q_j(\mathbf{u}, \mathbf{v}) \pm 2\mathrm{Re}\{iGg(x, y) e^{i\omega_{mn}t}\rho_{mn} \cos\omega t \cos kz\}, \quad j = m, n,$$
$$[\Gamma + \partial/\partial t + (\mathbf{u} + \mathbf{v})\nabla]\rho_{mn} =$$
$$= iGg(x, y) e^{-i\omega_{mn}t}(\rho_m - \rho_n) \cos\omega t \cos kz,$$
$$q_j(\mathbf{u}, \mathbf{v}) = Q_j W_M(\mathbf{u}) W_M(\mathbf{v}), \quad \rho_j = \rho_{jj}. \tag{3.2}$$

Here u + v is the total velocity of the atom, $\mathbf{u} \cdot \mathbf{k} = 0$, $\mathbf{v} \cdot \mathbf{k} = vk$. If we solve (3.2) under conditions (1.2) by successive approximations with respect to G^2, then the population can be regarded as independent of z, and we can seek a solution in the form (see[8])

$$\rho_j(x, y, z; t) = \rho_j(x, y),$$
$$\rho_{mn}(x, y, z; t) = \rho^+(x, y) e^{-i(\Omega t + kz)} + \rho^-(x, y) e^{-i(\Omega t - kz)}. \tag{3.3}$$

After substituting in (3.2), neglecting higher-order oscillations, we obtain in the resonant approximation

$$(\Gamma_j + ud/dx)\rho_j = q_j(u, v) \pm \mathrm{Re}\{iGg(x, y)(\rho^+ + \rho^-)\}, \quad j = m, n,$$
$$[\Gamma - i(\Omega \pm kv) + ud/dx]\rho^\pm = {}^1/_2 iGg(x, y)(\rho_m - \rho_n). \tag{3.4}$$

To simplify the calculations, the x axis is directed along u, so that the solution (3.4) depends on y parametrically.

The stationary generation power is proportional to the square of the field amplitude, which we determine from the equation for the energy balance:

$$G^2 \int dy \int dx \langle a(x, y; u, v)\rangle_{u,v} - G^4 \int dy \int dx \langle b(\Omega, x, y; u, v)\rangle_{u,v}$$
$$= G^2 \int dy \int dx R g^2(x, y). \tag{3.5}$$

On the left of the equal sign is the expression for the work of the field per unit time, and on the right are the integral losses in the resonator mirrors; R is the loss coefficient; a(x, y; u, v) and b(Ω, x, y; u, v) are the coefficients in the expansion of the work of the field in powers of G^2. We introduce the notation

$$\beta(\Omega, y; u) = \int dx \langle b(\Omega, x, y; u, v)\rangle_v \Big/ \int dx \int dy \langle a(x, y; u, v)\rangle_{u,v},$$

$$\beta(\Omega, u) = \int dy \beta(\Omega, y; u), \quad \beta(\Omega) = \langle \beta(\Omega, u)\rangle_u. \tag{3.6}$$

Near the generation threshold we obtain from (3.5)

$$G^2 = \frac{\eta - 1}{\beta(\Omega)}, \quad \eta = \frac{\int dy \int dx \langle a(x, y; u, v)\rangle_{u,v}}{R \int dy \int dx g^2(x, y)}; \tag{3.7}$$

is the excess of the excitation over the threshold value, $1/\beta(\Omega)$ describes the Lamb "dip" of interest to us.

Let us assume that $d \gg \lambda = 1/k$ or $1/k\bar{v} \ll d/\bar{v}$, where d is the characteristic distance over which g(x, y) changes noticeably. It follows then from the analysis given in Sec. 2 that in the case of homogeneous excitation for the field-work term linear in G^2 we can use formula (2.4), in which we replace G^2 by $G^2 g^2(x, y)$. Using also (1.2), we readily obtain from (3.4)

[1] The ρ_{ij} are normalized, as usual, to the number of atoms per unit volume and per unit velocity interval, unlike in [6,7], where they consider the number of atoms of given velocity crossing per unit time a surface of unit area.

$$\beta(\Omega, y; u)=\frac{N(u)}{2\alpha u^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} d\xi_1 \int_{-\infty}^{\xi_1} d\xi_2 g(x,y)g(\xi_1,y)g(\xi_2,y)$$
$$\times g(\xi_1+\xi_2-x,y)\exp\{-2\Gamma(x-\xi_1)/u\}(1+\cos[2\Omega(x-\xi_1)/u])\cdot$$
$$\times(\exp\{-\Gamma_m(\xi_1-\xi_2)/u\}+\exp\{-\Gamma_n(\xi_1-\xi_2)/u\}), \quad (3.8)$$
$$\alpha=\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx N_0 g^2(x,y).$$

Let us discuss the derived formula. When $u \ll \Gamma d$, $\Gamma_j d$, the exponentials together with the cosine can be regarded as δ-functions, and then

$$\beta(\Omega, u)=\frac{N(u)}{4a\Gamma}\left(\frac{1}{\Gamma_m}+\frac{1}{\Gamma_n}\right)\left(1+\frac{\Gamma^2}{\Gamma^2+\Omega^2}\right)\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx g^4(x,y), \quad (3.9)$$

in full agreement with the known "homogeneous" problem, where it is assumed that $v \ll \Gamma d$, $\Gamma_j d$. In the opposite limiting case when $u \gg \Gamma d$, Ωd we have

$$\beta(\Omega, u)=\frac{2N(u)}{au^2}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} d\xi_1 \int_{-\infty}^{\xi_1} d\xi_2 g(x,y)g(\xi_1,y)g(\xi_2,y)$$
$$\times g(\xi_1+\xi_2-x,y). \quad (3.10)$$

The results (3.9) and (3.10) can be easily interpreted in light of the statements made in Sec. 2. The first result means that the oscillations of the slow atom have time to "build up" over a distance much smaller than d, and by the same token the atom follows the values of the field at each point (x, y). In the second case, the atom does not have time to "build up," and passes through the entire localization region, i.e., formula (3.10) is the analog of (2.6). Of course, a slow atom interacts more effectively with the field than a fast one.

Let us consider the case $\Gamma \ll 1/\bar{\tau} = \bar{v}/d$ (long-lived systems). Using (3.9) and (3.10), we can estimate, in order of magnitude, the effective velocity interval that makes the main contribution to $\beta(\Omega)$. We start from the crude formula

$$\beta(\Omega) \sim 2\int_0^{\Gamma d} \beta^{(1)}(\Omega, u)\,du + 2\int_{\Gamma d}^{\infty} \beta^{(2)}(\Omega, u)\,du = \beta^{(1)}(\Omega)+\beta^{(2)}(\Omega), \quad (3.11)$$

where $\beta^{(1)}(\Omega, u)$ and $\beta^{2}(\Omega, u)$ are taken respectively from (3.9) and (3.10). We put for simplicity $\Gamma_m = \Gamma_n = \Gamma$ and $\Omega = 0$, and we consider two cases.

One-dimensional limitation of the field: g(x, y) is bounded in x, and the condition of the "homogeneous" problem is satisfied along y. The distribution with respect to the velocities u is then $W_M(u) = \exp\{-u^2/\bar{v}^2\}/\bar{v}\sqrt{\pi}$. After elementary calculations we obtain from (3.11)

$$\beta^{(1)}(0) \sim \bar{\tau}/\Gamma, \quad \beta^{(2)}(0) \sim \bar{\tau}//\Gamma. \quad (3.12)$$

For $\beta^{(1)}(0)$, the velocity interval is Γd. We determine the effective velocity interval for $\beta^{(2)}(0)$. Each atom with velocity $u > \Gamma d$ yields $\beta^{(2)}(0, u) \sim d^2/u^2$. With such a sharp decrease with respect to u, we can replace u by Γd, and then $\beta^{(2)}(0) \sim \Delta u/\Gamma^2\bar{v}$, so that $\Delta u \sim \Gamma d$. Thus, for $\beta(0) \sim \beta^{(1)}(0) + \beta^{(2)}(0)$ the effective velocity interval turns out to be of the order of Γd. If we recognize that for such atoms the time of the coherent interaction with the field is $\sim 1/\Gamma$ then we can state with assurance that in the one-dimensional case the width of the Lamb "dip" is determined by the natural line width Γ.

Two-dimensional limitation: g(x, y) is bounded in x and in y, with a characteristic dimension $d \ll \bar{v}/\Gamma$. The velocity distribution is in this case

$$W_M(u)=\frac{2u}{\bar{v}^2}\exp\{-u^2/\bar{v}^2\}. \quad (3.13)$$

From (3.11) we obtain

$$\beta^{(1)}(0) \sim \bar{\tau}^2, \quad \beta^{(2)}(0) \sim \bar{\tau}^2 \ln(1/\Gamma\bar{\tau}. \quad (3.14)$$

The contribution made to $\beta(\Omega)$ by atoms with velocities $u \le \Gamma d$, which is given by the term $\beta^{(1)}(0)$, turns out to be small. It is easy to verify that for $\beta^{(2)}(0)$ the effective velocity interval, although larger than Γd, is still much smaller than the mean-thermal velocity. Therefore the width of the Lamb "dip" in this case is still much smaller than $1/\bar{\tau}$.

An increase of the exponent of the velocity u (compared with (3.13)) in the velocity distributions (as in beam systems) causes the effective velocity interval to become $\sim \bar{v}$, and consequently the width of the Lamb "dip" is $\sim 1/\bar{\tau}$.

It is clear from the foregoing that the width of the Lamb "dip" will never be $\sim 1/\bar{\tau}$ in the case of volume excitation. In those cases it is determined by the value of Γ, and therefore greatest interest attaches to the conditions under which the inequality

$$\Gamma\bar{\tau} \ll 1 \quad (3.15)$$

is satisfied. When (3.15) is satisfied, the function $\beta(\Omega)$ changes significantly in the region

$$|\Omega| \ll 1/\bar{\tau}, \quad (3.16)$$

and therefore we shall henceforth assume that these conditions are satisfied.

From (3.8) we can discern the following remarkable property of the function $\beta(\Omega)$. If $\Gamma\bar{\tau} \ll 1$, the quantities $d/\beta(\Omega)$ and $d^2/\beta(\Omega)$ are proportional to the generation power in one-dimensional and two-dimensional limitation, and turn out to be practically invariant against the transformations $g(x, y) \to g(\lambda x, y)$ for the one-dimensional limitation and $g(x, y) \to g(\lambda x, \mu y)$ for the two-dimensional limitation. Thus, the generation power in the case of homogeneous excitation is practically independent of the dimension d of the light beam. This indicates that the decisive contribution to the generation power is made by atoms excited at a distance $D \sim \bar{v}/\Gamma \gg d$ outside the light beam. Indeed, from the results of the preceding section and from (3.8) it follows that at the same value of G the fast atoms are in the linear amplification regime, whereas the slow ones are already affected by saturation. In other words, under stationary conditions a given number of fast atoms per unit volume intensifies the field to a greater degree than an equal number of slow ones[2].

We note that the picture is somewhat different in the distribution of the characteristics of the radiation with respect to the longitudinal velocities v. All the atoms interacting with the field, i.e., atoms with resonant velocities, intensify the field to an equal degree and are saturated to an equal degree.

[2] If we normalize to the particle flux (see footnote [1]), then the probability of radiation per atom passing through the light beam turns out to be smaller for fast atoms than for slow ones. This is obvious, since a fast atom passing through the light beam changes the population less, and consequently, radiates less than a slow one.

4. LOCALIZATION OF EXCITATION

Usually it is always possible to separate the effective field region outside of which the field is practically equal to zero. It is of interest then to determine the frequency dependence of the generation power for atoms excited outside the effective range of the field. Such a formulation of the problem corresponds to simultaneous localization of the field and of the excitation.

Let us define g(x, y) in the following manner:

$$g(x,y)=\begin{cases} g(x,y)\neq 0 \text{ for } x_1(y)\leqslant x\leqslant x_2(y);\ y_1\leqslant y\leqslant y_2; \\ 0 \text{ for the remaining } x \text{ and } y. \end{cases} \quad (4.1)$$

We assume further that the excitation differs from zero and is homogeneous in the region where g(x, y) ≠ 0. For atoms with velocity u > 0, Eqs. (3.4) hold, with boundary conditions

$$\rho_j(x_1)=0,\ j=m,n. \quad (4.2)$$

The localization of the excitation influences already the field-work term linear in G^2. This is connected with the fact that the distribution of the unsaturated population with respect to the velocities u becomes essentially non-Maxwellian. Indeed, for the population difference at the point x with velocity u we have

$$N(x,u)=\left[\frac{Q_m}{\Gamma_m}(1-\exp\{-\Gamma_m(x-x_1)/u\})\right.$$
$$\left.-\frac{Q_n}{\Gamma_n}(1-\exp\{-\Gamma_n(x-x_1)/u\})\right]W_M(u). \quad (4.3)$$

For each x, the population difference at $u > \Gamma_m(x - x_1)$, $\Gamma_n(x - x_1)$ decreases like 1/u. Thus, the number of fast excited atoms turns out to be much smaller than that of the slow ones. This is obvious, since the fast atoms spend less time in the region where the excitation acts. It is easy to note that this additional discrimination of the fast atoms leads to an even greater narrowing of the spectral structures.

As a result of calculations analogous to those given in the preceding section, we obtain for $\beta(\Omega, y; u)$

$$\beta(\Omega,y;u)=\frac{1}{2\alpha_{1,2}u^2}\int_{x_1}^{x_2}dx\int_{(x+x_1)/2}^{x_1}d\xi_1\int_{x-\xi_1+x_1}^{\xi_1}d\xi_2 g(x,y)g(\xi_1,y)g(\xi_2,y)$$
$$\times g(\xi_1+\xi_2-x,y)\exp\{-2\Gamma(x-\xi_1)/u\}$$
$$\times(1+\cos[2\Omega(x-\xi_1)/u](\exp\{-\Gamma_m(\xi_1-\xi_2)/u\}$$
$$+\exp\{-\Gamma_n(\xi_1-\xi_2)/u\})N(\xi_1+\xi_2-x,u);$$
$$\alpha_{1,2}=\int_{y_1}^{y_2}dy\int_{x_1}^{x_2}dx\langle N(x,u)\rangle_u g^2(x,y), \quad (4.4)$$

where α_1 and α_2 correspond to the one-dimensional and two-dimensional limitations. These values are different because of the difference in the distributions with respect to the velocities u.

For $u \ll \Gamma(x_2 - x_1)$, $\Gamma_j(x_2 - x_1)$ we get from (4.4)

$$\beta(\Omega,u)=\frac{N_0W_M(u)}{4\alpha_{1,2}\Gamma}\left(\frac{1}{\Gamma_m}+\frac{1}{\Gamma_n}\right)\left(1+\frac{\Gamma^2}{\Gamma^2+\Omega^2}\right)\int_{y_1}^{y_2}dy\int_{x_1}^{x_2}dx g^4(x,y). \quad (4.5)$$

The frequency dependence here is exactly the same as in (3.9). In the case of the inverse inequality $u > \Gamma(x_2$ $u > \Gamma(x_2 - x_1)$ we have

$$\beta(\Omega,u)=\frac{2(Q_m-Q_n)W_M(u)}{\alpha_{1,2}u^3}$$
$$\times\int_{y_1}^{y_2}dy\int_{x_1}^{x_2}dx\int_{(x+x_1)/2}^{x}d\xi_1\int_{x-\xi_1+x_1}^{\xi_1}d\xi_2(\xi_1+\xi_2-x)g(x,y)g(\xi_1,y)g(\xi_2,y)$$
$$\times g(\xi_1+\xi_2-x,y). \quad (4.6)$$

Compared with (3.10), there appears here an additional power of u in the denominator—a reflection of the discrimination of the fast atoms in the population. Mathematically this means that upon averaging over the velocities u the Maxwellian exponential can be replaced by unity for both the one-dimensional and the two-dimensional limitations. Physically this fact denotes that the effective velocity interval in both cases is $\sim \Gamma d$, where $d = (x_2 - x_1)$ for one-dimensional limitation and $d = \max\{(x_2 - x_1), (y_2 - y_1)\}$ for two-dimensional limitation. Naturally, the width of the Lamb "dip" is then determined only by the value of Γ.

We thus arrive at an important conclusion. At any geometrical configuration of the field E(x, y) in the resonator, but with simultaneous localization of the field and of the excitation, the width of the fine spectral structures in the radiation characteristics should be determined by the natural line width Γ. For the validity of this statement, it suffices to satisfy the condition $d \gg \lambda/2\pi = 1/k$.

From the foregoing follows also a practical recommendation for obtaining ultranarrow spectral structures. Namely, it is necessary to strive to have the atoms excited only within the light beam. Moreover, with further concentration of the excitation in the vicinity of the axis of the light beam, the spectral structures have a tendency toward additional narrowing. This can be readily understood from the following simple reasoning. Let the excitation be localized in the vicinity $x_1 \leq x \leq x_2$ near the axis of a light beam with the dimension $d > (x_2 - x_1)$ along the x axis. The interaction with the field in the interval (x_1, x_2) would give spectral structures described by formula (4.4). The presence of a field outside the excitation region causes certain atoms that are "fast" relative to the dimension $(x_2 - x_1)$ to be still "slow" relative to the dimension d. By the same token, the time of coherent interaction is effectively increased, and this leads, on the whole, to a narrowing of the spectral structures.

In the case of localization of the excitation, the generation power becomes dependent on the transverse dimension d of the light beam. This is natural, since the total number of excited atoms depends on the dimension d.

So far we have discussed the general laws without specifying concretely the form of g(x, y). We present below calculations for certain concrete field configurations in the light beam.

5. RECTANGULAR FIELD PROFILE. HOMOGENEOUS EXCITATION

We specify g(x, y) in the form

$$g(x,y)=\begin{cases} 1 \text{ for } -x_0(y)\leqslant x\leqslant x_0(y),\ -y_0\leqslant y\leqslant y_0, \\ 0 \text{ for the remaining } x \text{ and } y. \end{cases} \quad (5.1)$$

With such a field configuration, the result can be easily obtained in analytic form. In addition, in certain real systems it is apparently possible to use such a model.

As a result of integration of (3.8) we obtain

$$\beta(\Omega,y;u)=\frac{x_0W_M(u)}{2S}\left\{\frac{1}{\Gamma\Gamma_m}\left(1+\frac{\Gamma^2}{\Gamma^2+\Omega^2}\right)\right.$$

$$+\frac{u}{2x_0}\Big[\frac{\Gamma_m^2\exp\{-2\Gamma x_0/u\}-\Gamma^2\exp\{-2\Gamma_m x_0/u\}}{\Gamma_m^2\Gamma^2(\Gamma-\Gamma_m)}$$
$$+\mathrm{Re}\frac{\Gamma_m^2\exp\{-2(\Gamma+i\Omega)x_0/u\}-(\Gamma+i\Omega)^2\exp\{-2\Gamma_m x_0/u\}}{\Gamma_m^2(\Gamma+i\Omega)^2(\Gamma+i\Omega-\Gamma_m)}\Big]$$
$$+\text{terms with the substitution }\Gamma_m\to\Gamma_n\Big\},\qquad(5.2)$$

where S is the area of the transverse cross section of the beam.

The Lamb "hole" will be described by means of two quantities: γ, the half-width at half-depth, obtained from the formula

$$\frac{1}{\beta(\infty)}-\frac{1}{\beta(\gamma)}=\frac{1}{2}\Big(\frac{1}{\beta(\infty)}-\frac{1}{\beta(0)}\Big),\qquad(5.3)$$

and ϑ, the relative second derivative at $\Omega = 0$. The quantity ϑ characterizes the curvature of the "dip" at its center and the reliability of the adherence to the transition frequency ω_{mn} in frequency-stabilization systems.

Let us consider first one-dimensional limitation of the field, i.e., we assume that $\Gamma y_0 \gg \bar{v}$. Elementary calculations yield

$$\beta(\Omega)=\frac{\bar\tau}{4\sqrt{\pi}}\Big\{\frac{\ln(\Gamma/\Gamma_m)}{\Gamma-\Gamma_m}+\frac{\ln(\Gamma/\Gamma_n)}{\Gamma-\Gamma_n}$$
$$+\mathrm{Re}\Big[\frac{\ln[(\Gamma+i\Omega)/\Gamma_n]}{\Gamma+i\Omega-\Gamma_m}+\frac{\ln[(\Gamma+i\Omega)/\Gamma_n]}{\Gamma+i\Omega-\Gamma_n}\Big]\Big\},\quad\bar\tau=\frac{2x_0}{\bar v},\qquad(5.4)$$

$\bar\tau$ determines the fraction of the saturated atoms (common factor), but does not enter in the frequency dependence at all. For the case $\Gamma = \Gamma_m = \Gamma_n$ we obtain a specially simple formula:

$$\beta(\Omega)=\frac{1}{2\sqrt{\pi}}\frac{\bar\tau}{\Gamma}\Big(1+\frac{\Gamma}{\Omega}\operatorname{arctg}\frac{\Omega}{\Gamma}\Big).\qquad(5.5)$$

The specific feature of the localization of the field becomes manifest in a different law governing the decrease at large Ω (Ω^{-1} instead of Ω^{-2} in the "homogeneous" problem). The parameters γ and ϑ turned out to be

$$\gamma=4\Gamma,\quad\vartheta=1/6\Gamma^2.\qquad(5.6)$$

For comparison, we present the values of these parameters in the "homogeneous" problem:

$$\gamma=\sqrt{2}\Gamma,\quad\vartheta=1/2\Gamma^2.\qquad(5.7)$$

Let us proceed to the two-dimensional limitation of the field. To simplify the calculations we assume that the light beam has a circular cross section of radius r. Leaving only the principal terms, we arrive at the following result:

$$\beta(\Omega)=\frac{\bar\tau^2}{16}\sum_{j=m,n}\Big\{\frac{\Gamma_j\ln\Gamma_j\bar\tau-\Gamma\ln\Gamma\bar\tau}{\Gamma-\Gamma_j}\qquad(5.8)$$
$$+\mathrm{Re}\frac{\Gamma_j\ln\Gamma_j\bar\tau-(\Gamma+i\Omega)\ln(\Gamma+i\Omega)\bar\tau}{\Gamma+i\Omega-\Gamma_j}\Big\}\quad\bar\tau=\frac{2r}{\bar v}.$$

For $\Gamma_m = \Gamma_n = \Gamma$ the formula is more lucid:

$$\beta(\Omega)=\frac{\bar\tau^2}{8}\Big(\ln\frac{1}{\Gamma\bar\tau}+\ln\frac{1}{\sqrt{\Gamma^2+\Omega^2\bar\tau}}\Big),\qquad(5.9)$$
$$\gamma=(\Gamma/\bar\tau^2)^{1/3},\quad\vartheta=\Big(\Gamma^2\ln\frac{1}{\Gamma\bar\tau}\Big)^{-1}.$$

When the conditions $\Gamma\bar\tau \ll 1$ are used, the width of the "dip" is much smaller than $1/\bar\tau$. In the customarily encountered line shapes (Gaussian or dispersion shape) the line width is connected with the second derivative at the center by the relation $\vartheta \approx \gamma^{-2}$. In this case this relation is violated. The parameter ϑ is determined practically by the value of Γ, just as in the "homogeneous" and one-dimensional problems, i.e., it is much larger than γ^{-2}. This means that the plot of $1/\beta(\Omega)$ at the center is much steeper than expected on the basis of the value of γ.

According to (5.9), two-dimensional limitation leads to a broadening of the "hole" in comparison with the one-dimensional limitation, this being connected with the decrease of the relative fraction of the slow atoms (the factor $u/\bar v$ in the Maxwell distribution).

6. RECTANGULAR FIELD PROFILE. LOCALIZED EXCITATION

The dependence of the field on the coordinates is retained in the form (5.1), and the excitation is assumed to be homogeneous and different from zero only in the region of action of the field. Starting from (4.3) and (4.4) and duplicating, with small variations, the procedure in the preceding section, we obtain the following results for the case of $\Gamma_m = \Gamma_n = \Gamma$ and one-dimensional limitation:

$$\alpha_1=\bar\tau S(Q_m-Q_n)\ln\frac{1}{\Gamma\bar\tau},\quad\beta(\Omega)=\frac{1}{4\Gamma^2\ln(1/\Gamma\bar\tau)}\Big[1+\frac{\Gamma^2}{\Omega^2}\ln\Big(1+\frac{\Omega^2}{\Gamma^2}\Big)\Big],$$
$$\gamma=2{,}5\Gamma,\quad\vartheta=1/4\Gamma^2;\qquad(6.1)$$

for two-dimensional limitation we have

$$\alpha_2=\frac{4\sqrt{\pi}}{3}r^2\bar\tau(Q_m-Q_n),\quad\beta(\Omega)=\frac{3\sqrt{\pi}}{64}\frac{\bar\tau}{\Gamma}\Big[1+\frac{2\Gamma}{\Omega}\operatorname{arctg}\frac{\Omega}{\Gamma}$$
$$-\frac{\Gamma^2}{\Omega^2}\ln\Big(1+\frac{\Omega^2}{\Gamma^2}\Big)\Big],$$
$$\gamma=6{,}5\Gamma,\quad\vartheta=1/12\Gamma^2.\qquad(6.2)$$

In accordance with the general conclusions of Sec. 4, the quantities γ and ϑ are determined by Γ and the difference from the "homogeneous" problem lies only in the coefficients.

Real fields and excitations have, naturally, a more complicated dependence on the coordinates x and y than that considered above. It is obvious, however, that this leads to a change of only the numerical coefficients in the parameters γ and ϑ. The results for the case of practical interest, that of a Gaussian profile of the light beam, obtained in the next section by numerical integration, verify the validity of this statement.

7. GAUSSIAN FIELD PROFILE. HOMOGENEOUS EXCITATION

In a resonator with confocal mirrors, the dependence of g(x, y) on the coordinates is of the form

$$g(x,y)=\exp(-x^2/a^2-y^2/b^2).\qquad(7.1)$$

Let us consider as before two cases:

1) $a\Gamma \ll v$, $b\Gamma \gg v$, corresponding to the one-dimensional limitation;

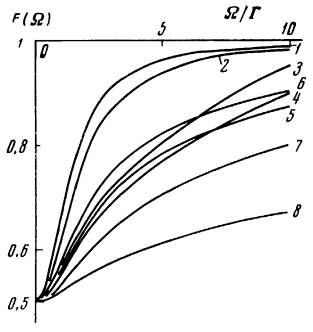
2) $a\Gamma \ll v$, $b\Gamma \ll v$—two-dimensional limitation.

From (3.8) we obtain

$$\beta(\Omega)=\frac{\sqrt{2}}{4}\frac{(\Gamma\bar\tau)^k}{\Gamma^2}\int_0^\infty d\xi\int_0^\infty d\eta\frac{e^{-2\xi}(1+\cos(2\Omega\xi/\Gamma))}{[(\Gamma\bar\tau)^2+\xi^2+(\xi+\eta)^2]^{k/2}}\cdot\qquad(7.2)$$
$$\times(\exp\{-\Gamma_m\eta/\Gamma\}+\exp\{-\Gamma_n\eta/\Gamma\}),\quad\bar\tau=a/\bar v;$$

k = 1 and 2 corresponds respectively to one-dimensional and two-dimensional limitation, and in the latter case we assume for simplicity that a = b.

The results of the numerical integration are shown



Form of the Lamb "dip" in the generation power for certain light beams at different parameters $\Gamma\bar{\tau}$, $f(\Omega) = \beta(0)/2\beta(\Omega)$.

in the figure. For comparison, we show curve 1, which describes the Lamb "dip" in the usual "homogeneous" problem. Curves 2–5 correspond to one-dimensional limitation of the field and homogeneous excitation. Curves 4 and 5 are given for the limiting value of $\Gamma\bar{\tau} \to 0$ for Gaussian and rectangular light-beam profiles, respectively. We note that they differ insignificantly from each other. It can therefore be assumed that the structure of the Lamb "dip" generally depends little on the profile of the light beam.

At an arbitrary value of the parameter $\Gamma\bar{\tau}$, a plot of the function $F(\Omega) = \beta(0)/2\beta(\Omega)$ lies above 4 and 5, but below 1. For example, for a Gaussian profile, the curves 2 and 3 are given for the values $\Gamma\bar{\tau} = 1$ and $\Gamma\bar{\tau} = 10^{-1}$. In the case of two-dimensional limitation, the function $F(\Omega)$ gives the curves 7 (localized excitation, $\Gamma\bar{\tau} \to 0$) and 8 (homogeneous excitation, $\Gamma\bar{\tau} = 10^{-2}$). In all cases, at $\Gamma\tau \ll 1$, there is observed, besides the broadening of the "dip" a noticeable drop of the wings compared with the "homogeneous" problem.

For all the curves indicated above, it was assumed that $\Gamma_m = \Gamma_n = \Gamma$. If we neglect the collisions, as before, but assume that the lifetime at the levels m and n is different, then the "hole" at the same value of Γ becomes somewhat narrower (e.g., curve 6 for the rectangular profile, one-dimensional limitation, homogeneous excitation, and at $\Gamma_n/\Gamma_m = 10$).

8. ALLOWANCE FOR THE RECOIL EFFECT

It follows from the foregoing that the transit factor is not the principal limitation on the production of ultranarrow spectral structures. According to[8], for $\Gamma \lesssim \delta = k^2\hbar/2m \sim 10^4$ sec^{-1}, an important role may be played by the recoil effect in the emission and absorption of a photon. It is therefore of interest to consider the influence of recoil under the conditions of the present investigation.

Following[8], we take into account the spatial quantization by introducing the Wigner quantum distribution function. We make the same assumptions with respect to this function as in the derivation of (3.4). As a result we obtain the equations

$$(\Gamma_m + ud/dx) f_{mm}(v) = q_m(v) + \mathrm{Re}\{iGg(x, y)[f^+(v+\xi) + f^-(v-\xi)]\},$$
$$(\Gamma_n + ud/dx) f_{nn}(v) = q_n(v) - \mathrm{Re}\{iGg(x, y)[f^+(v-\xi) + f^-(v+\xi)]\},$$
$$[\Gamma - i(\Omega + kv) + ud/dx] f^+(v) = {}^1/_2 iGg(x, y)[f_{mm}(v-\xi) - f_{nn}(v+\xi)],$$
$$[\Gamma - i(\Omega - kv) + ud/dx] f^-(v) = {}^1/_2 iGg(x, y)[f_{mm}(v+\xi) - f_{nn}(v-\xi)], \quad (8.1)$$

f_{mm} and f_{nn} are the Wigner-matrix elements independent of z, $\xi = \hbar k/2m$ is the change in velocity upon emission or absorption of a photon of frequency $\omega = kc$. The remaining notation is the same as in the preceding sections. As $\hbar \to 0$, Eqs. (8.1) go over, as they should, into (3.4).

To determine the amplitude of the steady-state field near the generation threshold it is necessary to start not from (3.7), but from the formula (see[8]):

$$G^2 = \frac{\eta - 1 - [(\Omega + \Delta)/k\bar{v}]^2}{\beta(\Omega)}, \quad \Delta = \delta(N_m + N_n)/(N_m - N_n), \quad (8.2)$$

where N_m and N_n are the values of the unsaturated populations of the levels m and n.

In the case of homogeneous excitation we have for $\beta(\Omega, y; u)$

$$\beta(\Omega, y; u) = \frac{N(u)}{2au^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} d\xi_1 \int_{-\infty}^{\xi_1} d\xi_2 g(x, y) g(\xi_1, y) g(\xi_2, y) g(\xi_1 + \xi_2 - x, y) \cdot$$
$$\times e^{-2\Gamma(x-\xi_1)/u}\left\{\exp\left[-\frac{\Gamma_m}{u}(\xi_1 - \xi_2)\right]\left(1 + \cos\frac{2(\Omega+\delta)}{u}(x - \xi_1)\right)\right.$$
$$\left. + \exp\left[-\frac{\Gamma_n}{u}(\xi_1 - \xi_2)\right]\left(1 + \cos\frac{2(\Omega-\delta)}{u}(x - \xi_1)\right)\right\}. \quad (8.3)$$

This formula differs from (3.8) in that the arguments of the cosines contain not Ω but $\Omega \pm \delta$, and each of the cosines is multiplied only by one of the exponentials $\exp\{-\Gamma_j(\xi_1 - \xi_2)/u\}$; j = m, n. As a result, the function $\beta(\Omega)$ has two maxima (at $\Omega = \pm\delta$) instead of one. The distance between the maxima is 2δ. In addition, the frequency dependence and the value at the maximum of the cosine terms are determined by the lifetime at the corresponding level. The actual calculation of $\beta(\Omega)$ is simply a repetition of the preceding operations.

The picture is somewhat different in the case of localization of the excitation together with the field. The difference in the distribution with respect to the rates of populations of the upper and lower level causes the form and the height of each of the "peaks" to be determined by the lifetimes at both levels.

Upon localization of excitation together with the field, the width of the Lamb "dip" is minimal and is determined by the quantities Γ, Γ_m, and Γ_n. In this case it is necessary to expect the maximum resolution of two "dips." For example, for $\delta \sim 10^4$ sec^{-1} and $\Gamma \sim 10^3$ sec^{-1}, the "dips" practically do not overlap, i.e., the recoil effect is easily observed. In the case of frequency stabilization, it is necessary to take into account the splitting of the "dip" with a high degree of accuracy.

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